

# PROOF-THEORETIC SEMANTICS FOR DYNAMIC LOGICS

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6 October 2015

# PROOF-THEORETIC SEMANTICS

Theories of meaning	
Denotational (model-theoretic)	Inferential (proof-theoretic)
<u>Tarski</u> : Meaning is out there	<u>Gentzen</u> : Meaning is in <b>RULES</b>

- ▶ Wittgenstein: meaning is use (very influential in philosophy of language)
- ▶ Wansing: meaning is **correct** use!
- ▶ not all proof systems are good environments for an inferential theory of meaning.

# GOOD PROOF SYSTEMS FOR DLs: DESIDERATA

- ▶ An **independent** account of dynamic logics:
  - ▶ Proof-theoretic semantic approach;
- ▶ Intuitive, **user-friendly** rules;
- ▶ **Good performances**:
  - ▶ soundness & completeness,
  - ▶ cut-elimination & sub-formula property,
  - ▶ decidability.
- ▶ A **modular** account of dynamic logics:
  - ▶ charting the space of DLs by adding/subtracting rules,
  - ▶ transfer of results with minimal changes.

# PROBLEMS: THE CASE STUDY OF DEL

$$\langle \alpha \rangle p \leftrightarrow Pre(\alpha) \wedge p$$

$$\langle \alpha \rangle (A \vee B) \leftrightarrow \langle \alpha \rangle A \vee \langle \alpha \rangle B$$

$$\langle \alpha \rangle \neg A \leftrightarrow Pre(\alpha) \wedge \neg \langle \alpha \rangle A$$

$$\langle \alpha \rangle \langle a \rangle A \leftrightarrow Pre(\alpha) \wedge \bigvee \{ \langle a \rangle \langle \beta \rangle A \mid \alpha a \beta \}$$

1. not closed under **uniform substitution**;
2. use of **meta-linguistic abbreviation**  $Pre(\alpha)$ ;
3. use of **labels**  $\alpha a \beta$ .

# THE CASE STUDY OF PDL

$$[\alpha](A \rightarrow B) \rightarrow ([\alpha]A \rightarrow [\alpha]B)$$

$$[\alpha \cup \beta]A \leftrightarrow [\alpha]A \wedge [\beta]A$$

$$[\alpha ; \beta]A \leftrightarrow [\alpha][\beta]A$$

$$[?A]B \leftrightarrow (A \rightarrow B)$$

$$[\alpha](A \wedge B) \leftrightarrow [\alpha]A \wedge [\alpha]B$$

$$[\alpha^*]A \leftrightarrow A \wedge [\alpha][\alpha^*]A$$

$$A \wedge [\alpha^*](A \rightarrow [\alpha]A) \rightarrow [\alpha^*]A$$

# DISPLAY CALCULI

- ▶ Natural generalization of sequent calculi;
- ▶ sequents  $X \vdash Y$ , where  $X, Y$  STRUCTURES:  
 $\phi$  ,  $\phi; \psi$  ...,  $X > Y, \dots$

- ▶ DISPLAY PROPERTY:

$$\frac{\frac{\frac{Y \vdash X > Z}{X; Y \vdash Z}}{Y; X \vdash Z}}{X \vdash Y > Z}$$

- ▶ display property: **adjunction** at the structural level.
- ▶ **Canonical proof of cut elimination**

# MORE ON STRUCTURAL CONNECTIVES

- ▶ One for two:

$>$		$;$		$ $		$\{a\}$		$\widehat{a}$		$\{\alpha\}$		$\widehat{\alpha}$	
$>$	$\rightarrow$	$\wedge$	$\vee$	$\top$	$\perp$	$\langle a \rangle$	$[a]$	$\widehat{a}$	$\underline{a}$	$\langle \alpha \rangle$	$[\alpha]$	$\widehat{\alpha}$	$\underline{\alpha}$

- ▶ Again, dynamic adjoints needed for display rules:

$$\frac{X \vdash \{a\}Y}{\widehat{a}X \vdash Y} \qquad \frac{\{a\}X \vdash Y}{X \vdash \widehat{a}Y}$$

$$\frac{X \vdash \{\alpha\}Y}{\widehat{\alpha}X \vdash Y} \qquad \frac{\{\alpha\}X \vdash Y}{X \vdash \widehat{\alpha}Y}$$

# THE MULTI-TYPE APPROACH

- ▶ Ag Act Fnc Fm;
  - ▶ no ancillary symbols; all types are **first-class citizens**;
- ▶ Additional expressivity:
  - ▶ operational connectives **merging different types**:

$$\Delta_1, \blacktriangle_1 : \text{Act} \times \text{Fm} \rightarrow \text{Fm} \quad \langle \alpha \rangle A \rightsquigarrow \alpha \Delta_1 A$$

$$\Delta_2, \blacktriangle_2 : \text{Ag} \times \text{Fm} \rightarrow \text{Fm} \quad \langle a \rangle A \rightsquigarrow a \Delta_2 A$$

$$\Delta_3, \blacktriangle_3 : \text{Ag} \times \text{Fnc} \rightarrow \text{Act}$$

- ▶ Modularity: by adding or subtracting types (Games, strategies, coalitions) one can chart the whole space of dynamic logics.

for  $1 \leq i \leq 3$ ,

	$\Delta_i$	$\blacktriangle_i$	$\neg \triangleright_i$	$\neg \blacktriangleright_i$
$\Delta_i$		$\blacktriangle_i$		$\neg \blacktriangleright_i$



# A GLIMPSE AT RULES FOR DEL

Single-type, first version: formulas as side conditions (and rules with labels);

$$\text{swap-in}_L \frac{\text{Pre}(\alpha); \{\alpha\}\{a\}X \vdash Y}{\text{Pre}(\alpha); \{a\}\{\beta\}_{\alpha a \beta} X \vdash Y}$$

Single-type, emended: purely structural (but labels still there);

$$\text{swap-in}'_L \frac{\{\alpha\}\{a\}X \vdash Y}{\Phi_\alpha; \{a\}\{\beta\}_{\alpha a \beta} X \vdash Y}$$

Multi-type: no side conditions and no labels.

$$\text{swap-in}_L \frac{a \blacktriangle_2(\alpha \blacktriangle_1 X) \vdash Y}{(a \blacktriangle_3 \alpha) \blacktriangle_1(a \blacktriangle_2 X) \vdash Y}$$

# A GLIMPSE AT RULES FOR PDL

$$\oplus\ominus \frac{\Pi^\oplus \vdash \Delta}{\Pi \vdash \Delta^\ominus}$$

$$\omega \Delta \frac{\left( \Pi^{(n)} \Delta_1 X \vdash Y \mid n \geq 1 \right)}{\Pi^\oplus \Delta_0 X \vdash Y}$$

# CANONICAL CUT ELIMINATION, 1/3

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined congruence:
  - ▶ same shape, same position, **same type**, non-proliferation;
3. **principal = displayed** (**Exception**: principal fma's in axioms)
  - ▶ Generaliz.: axioms are **closed** under display rules (when applicable);
4. rules are closed under **uniform substitution** of congruent parameters **within each type**;
5. **reduction strategy** exists when cut formulas are both principal.

## SPECIFIC TO MULTI-TYPE SETTING:

6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

**THM:** For any (multi-type) calculus satisfying list above, the cut elimination theorem can be proven.

# CANONICAL CUT ELIMINATION, 2/3

Two main cases + subcases.

(a) **Both cut formulas are principal.** by 5. (cut is either eliminated or “broken down” into cuts of lower rank).

(b) **At least one cut formula is parametric.** Subcase (b1):  $a_u$  principal in axiom. Then,

$$\begin{array}{c}
 \vdots \pi_1 \\
 x \vdash a \\
 \hline
 x \vdash y
 \end{array}
 \quad
 \begin{array}{c}
 (x' \vdash y')[a_u^{pre}, a_{suc}] \\
 \vdots \pi_2 \\
 a \vdash y
 \end{array}
 \quad
 \rightsquigarrow
 \quad
 \begin{array}{c}
 \vdots \pi_1 \\
 x \vdash a \\
 \hline
 x \vdash y''[a_{suc}]
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \pi'' \\
 (x' \vdash y')[x^{pre}, a_{suc}] \\
 \vdots \pi_2[x/a_u] \\
 \hline
 x \vdash y
 \end{array}$$

## CANONICAL CUT ELIMINATION, 3/3

Subcase (b2):  $a_u$  principal in other rule. Then,  $a_u$  is in display, and hence:

$$\frac{\frac{\frac{\vdots \pi_1}{x \vdash a} \quad \frac{\vdots \pi_2}{a \vdash y}}{x \vdash y}}{\vdots \pi'_2} \quad \frac{\frac{\frac{\vdots \pi_1}{x \vdash a} \quad \frac{\vdots \pi'_2}{a_u \vdash y'}}{x \vdash y'}}{\vdots \pi_2[x/a]} \rightsquigarrow$$

## CANONICAL CUT ELIMINATION, 3/3

Subcase (b2):  $a_u$  principal in other rule. Then,  $a_u$  is in display, and hence:

$$\begin{array}{c}
 \vdots \pi'_2 \\
 a_u \vdash y' \\
 \\
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{x \vdash a \quad a \vdash y}{x \vdash y}
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \vdots \pi_1 \quad \vdots \pi'_2 \\
 \frac{x \vdash a \quad a_u \vdash y'}{x \vdash y'} \\
 \\
 \vdots \pi_2[x/a] \\
 x \vdash y
 \end{array}$$

Subcase (b3):  $a_u$  parametric. Then:

$$\begin{array}{c}
 \vdots \pi'_2 \\
 (x' \vdash y')[a_u]^{pre} \\
 \\
 \vdots \pi_1 \quad \vdots \pi_2 \\
 \frac{x \vdash a \quad a \vdash y}{x \vdash y}
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \vdots \pi'_2 \\
 (x' \vdash y')[x/a_u^{pre}] \\
 \\
 \vdots \pi_2[x/a_u^{pre}] \\
 x \vdash y
 \end{array}$$

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