Event Structures and Games

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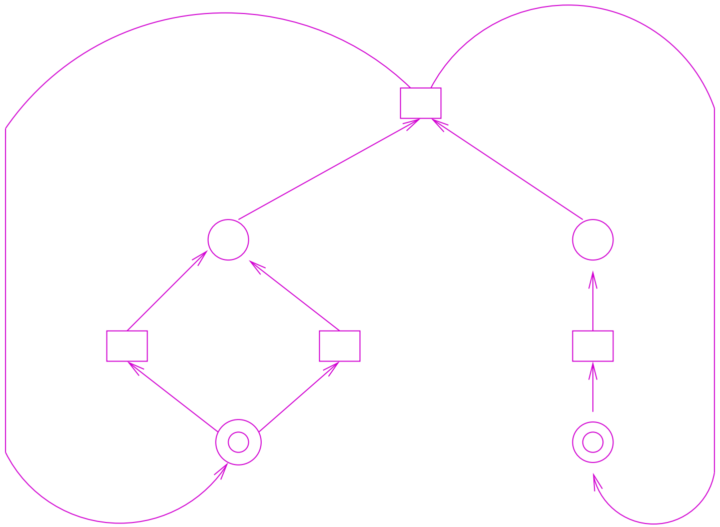
Strathclyde101, 26 November 2020

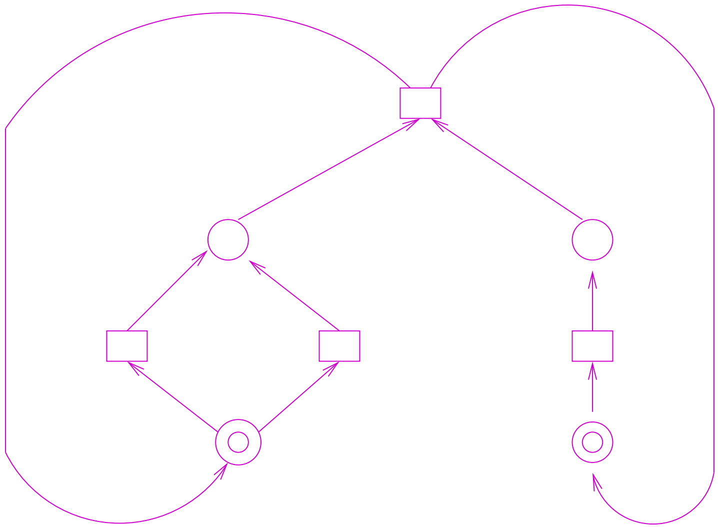
Event structures, a model based on causal dependency of events - concurrent analogue of trees. Locality $\rightsquigarrow$ causal dependency.
$\rightsquigarrow$ Broad applications, in security protocols, systems biology, weak memory, partial-order model checking, distributed computation, logic, semantics, ...

Distributed games, with behaviour based on event structures, rather than trees.
Aim: to generalise domain theory, tackle its anomalies and limitations w.r.t. concurrency and quantitative aspects; repair the divides between denotational vs. operational, semantic vs. algorithmic.
Ways to compose winning and optimal strategies $\rightsquigarrow$ *structural game theory.*

A (basic) Petri net



Unfolding a Petri net



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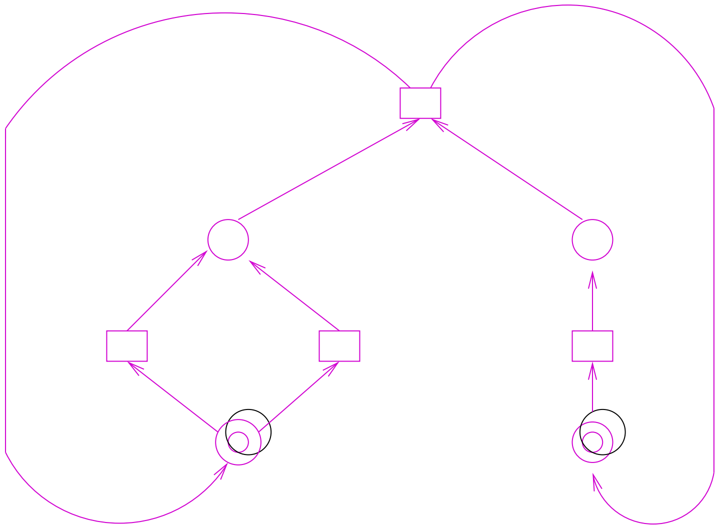

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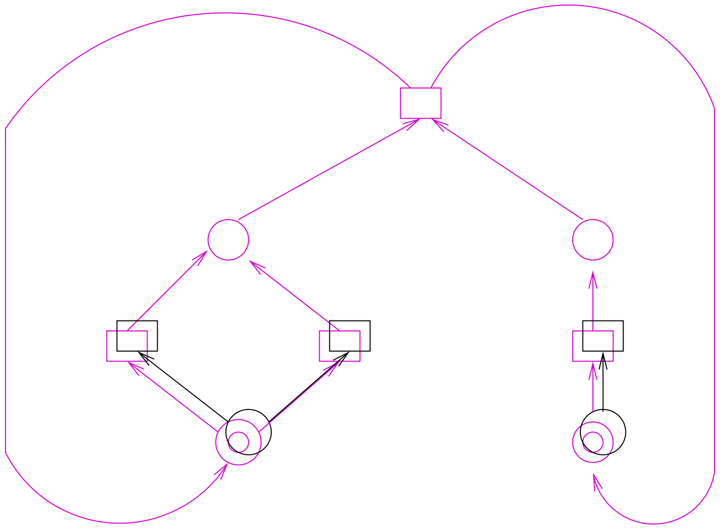

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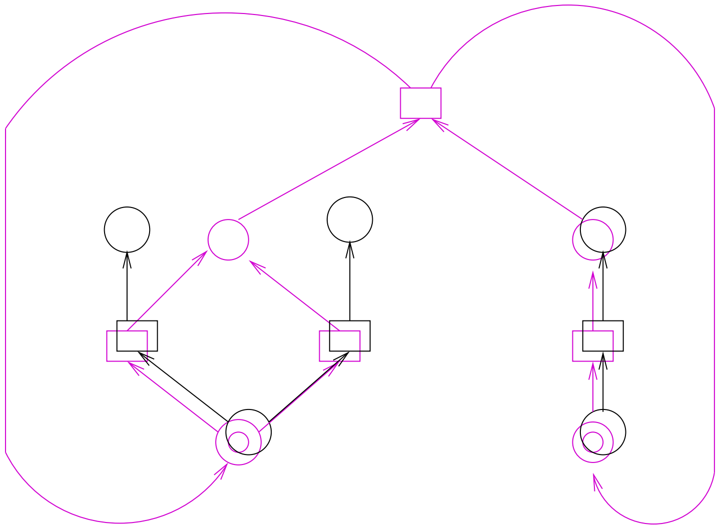

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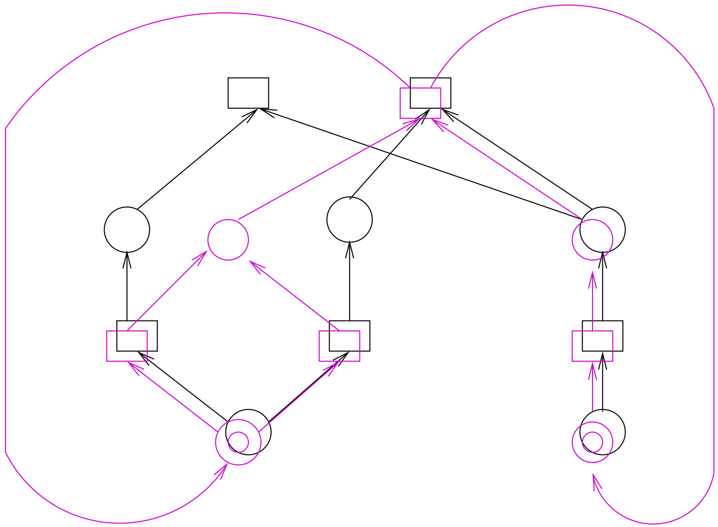

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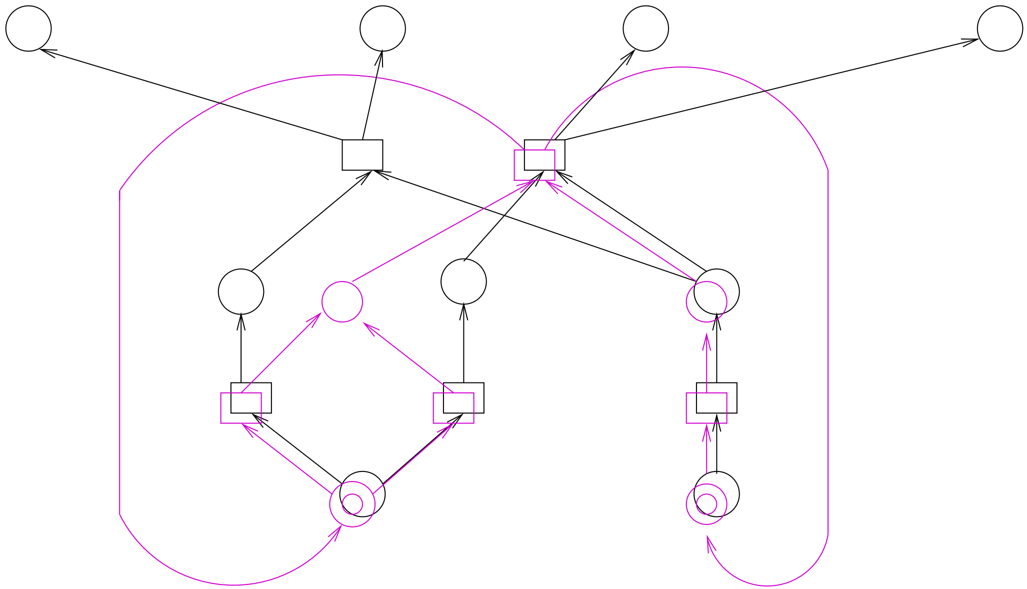

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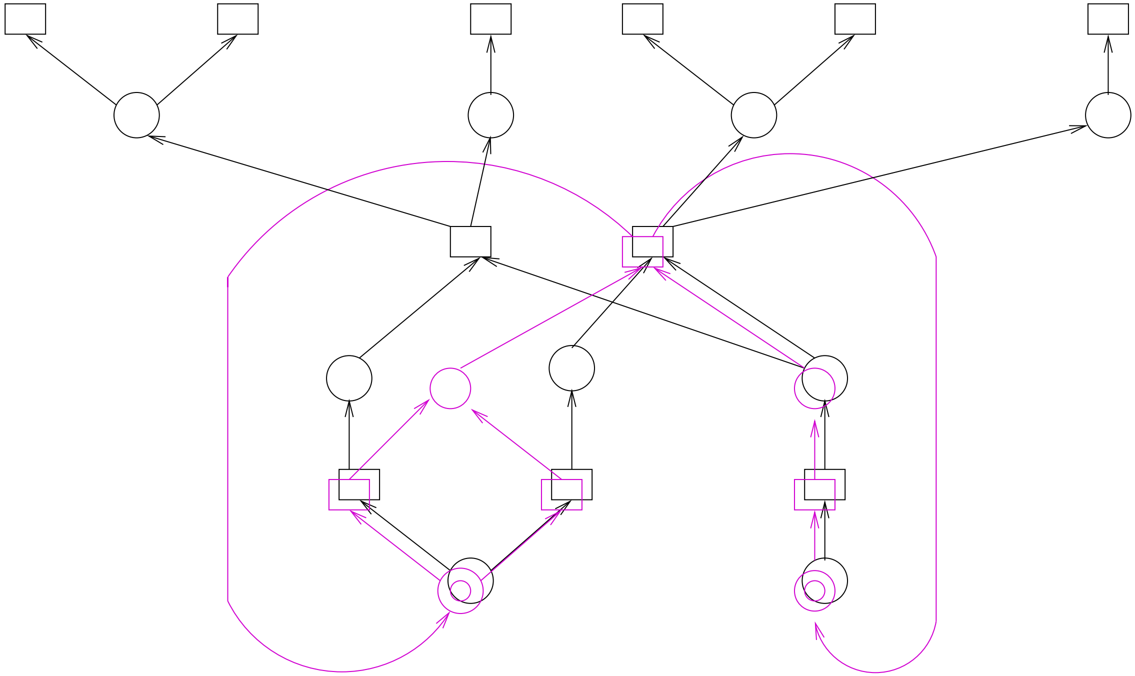

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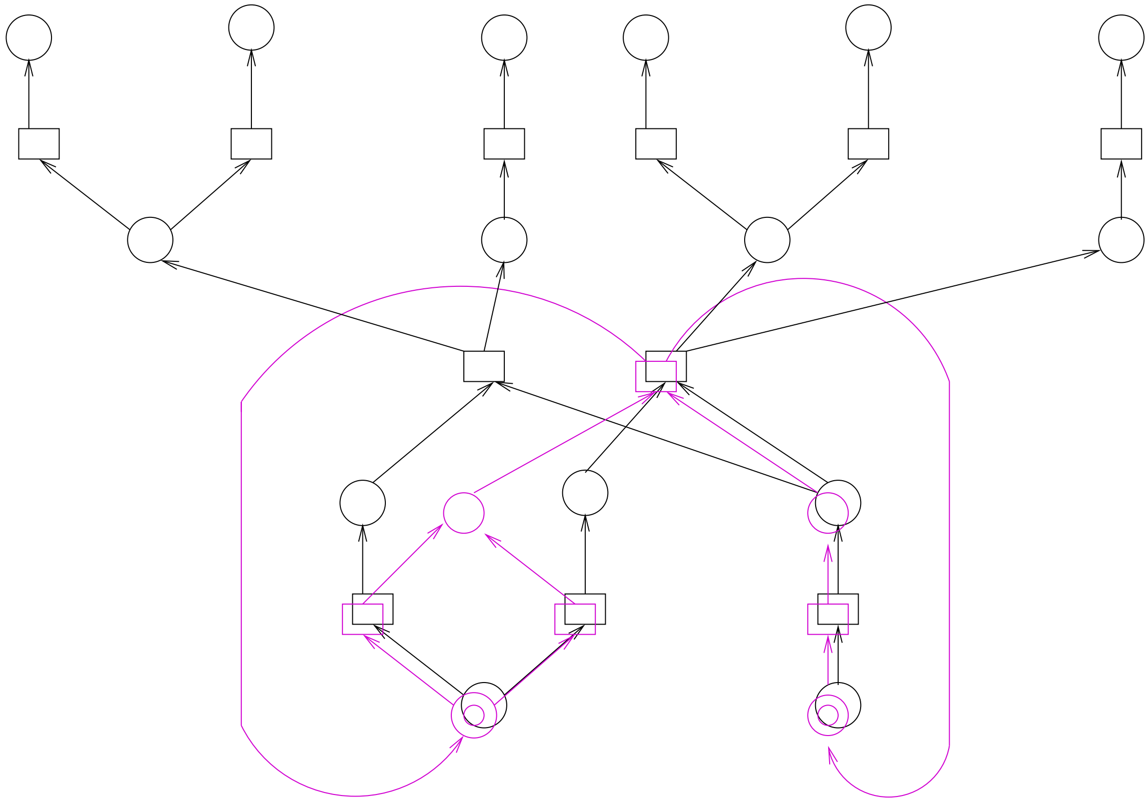

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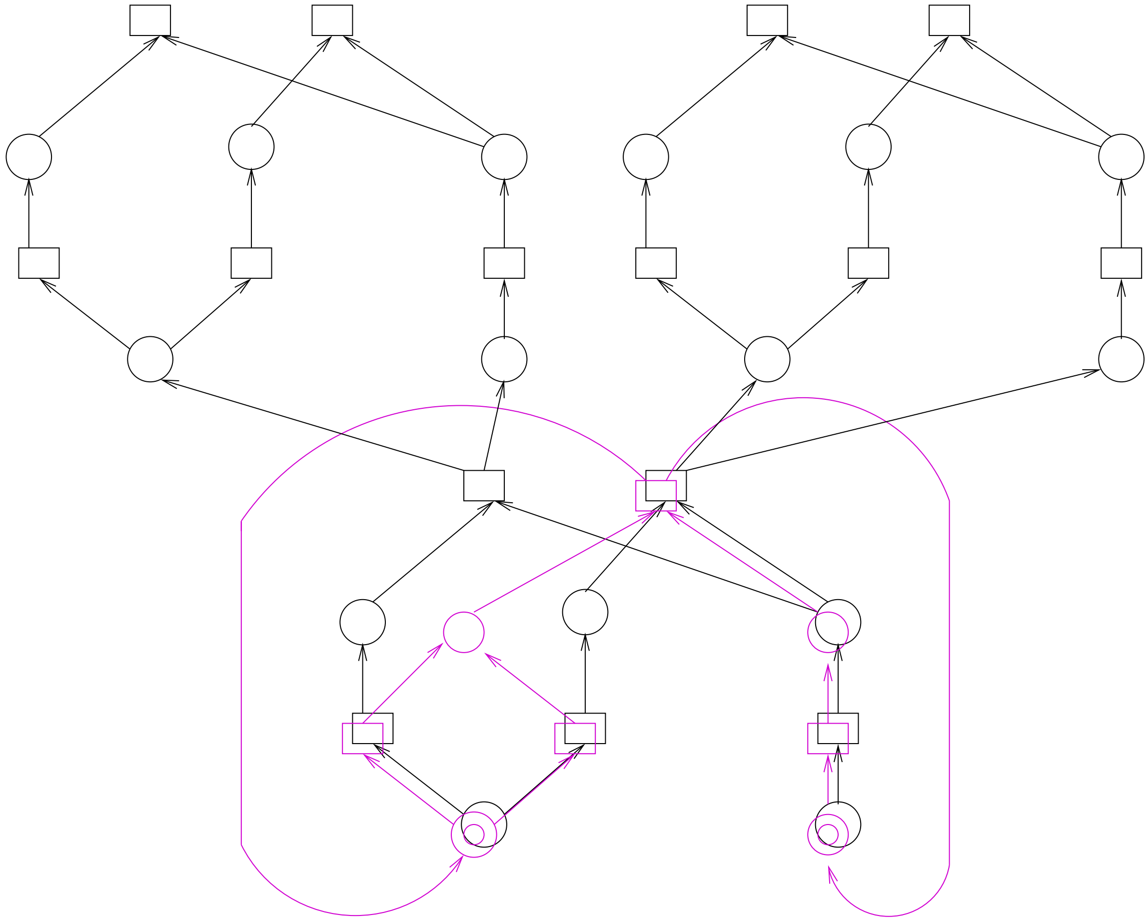

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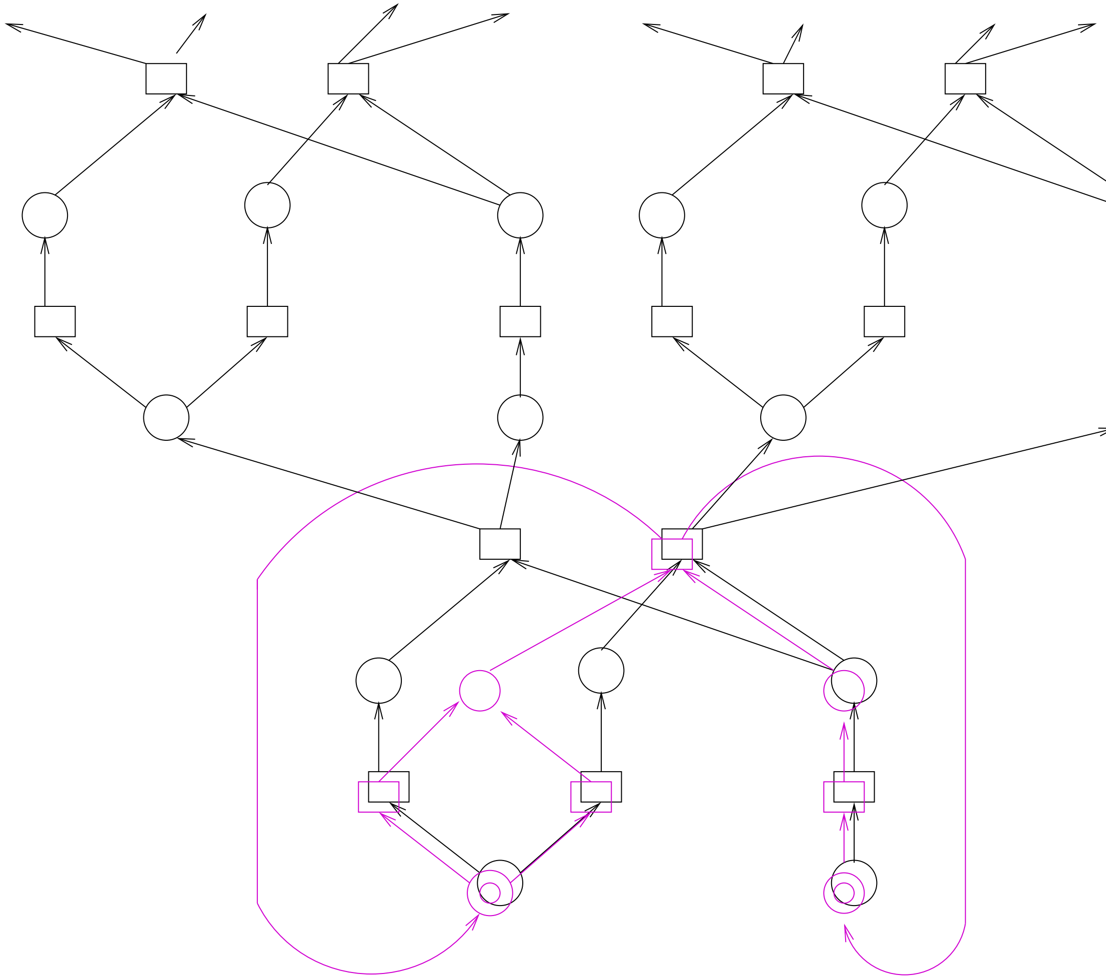
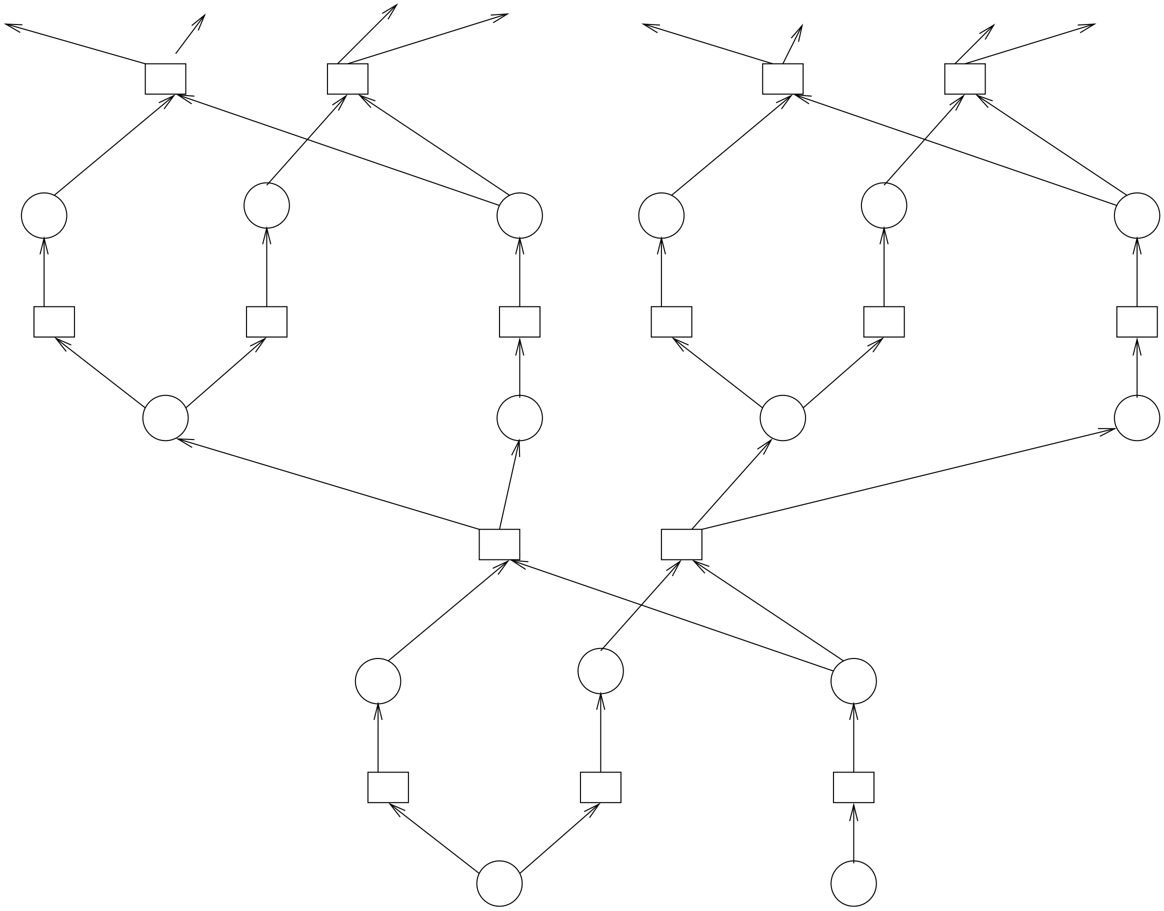
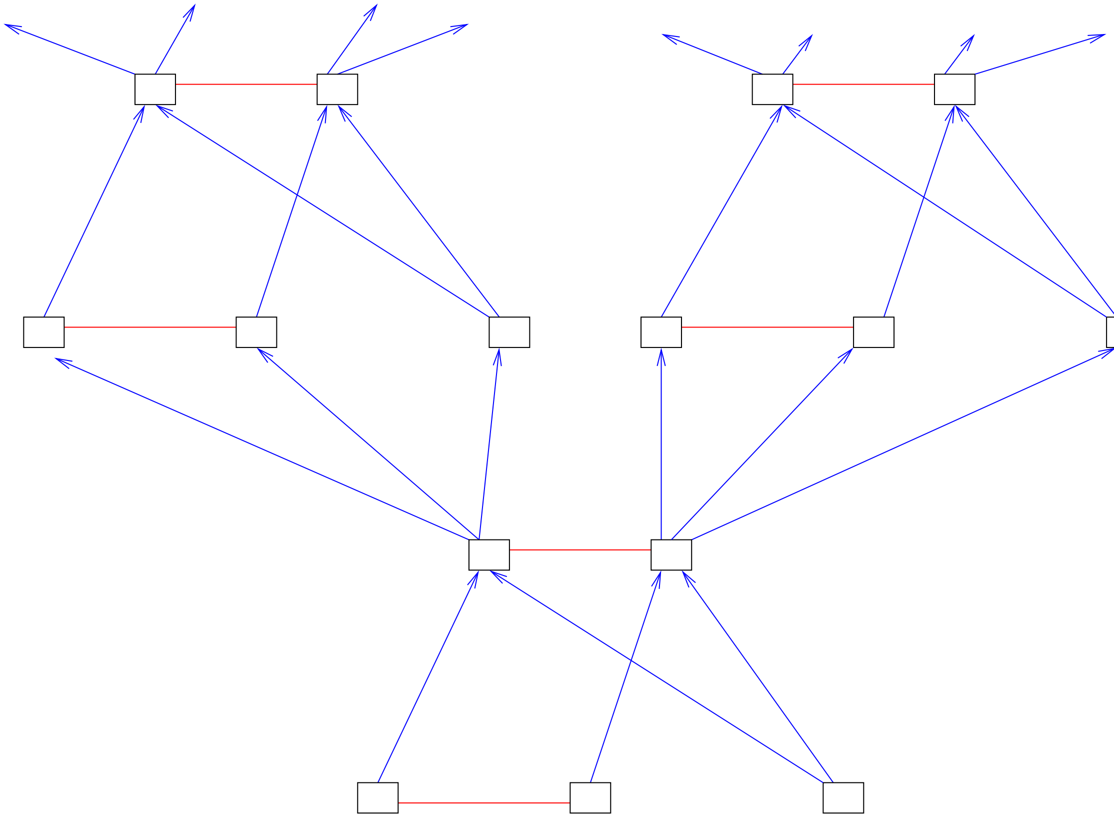


An occurrence net



An event structure



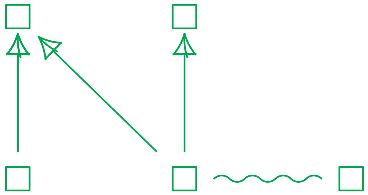
Event structures - the formal definition of the simplest kind

Definition

An event structure comprises $(E, \leqslant, \#)$, consisting of a set of events E
- partially ordered by $\leqslant$, the causal dependency relation, and
- a binary irreflexive symmetric relation, the conflict relation,

which satisfy $\{e' \mid e' \leqslant e\}$ is finite and $e \# e' \leqslant e'' \implies e \# e''$.

Two events are concurrent when neither in conflict nor causally related.



Definition

The finite configurations, $\mathcal{C}(E)$, of an event structure E consist of those finite subsets $x \subseteq E$ which are

Consistent: $\forall e, e' \in x.\ \neg(e \# e')$ and

Down-closed: $\forall e, e'.\ e' \leqslant e \in x \implies e' \in x$.

Maps of event structures

Definition

A map of event structures $f : E \to E'$
is a partial function on events $f : E \rightharpoonup E'$ such that

for all $x \in \mathcal{C}(E)$, $\quad fx \in \mathcal{C}(E')$ and

if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$, then $e_1 = e_2$. *(local injectivity)*

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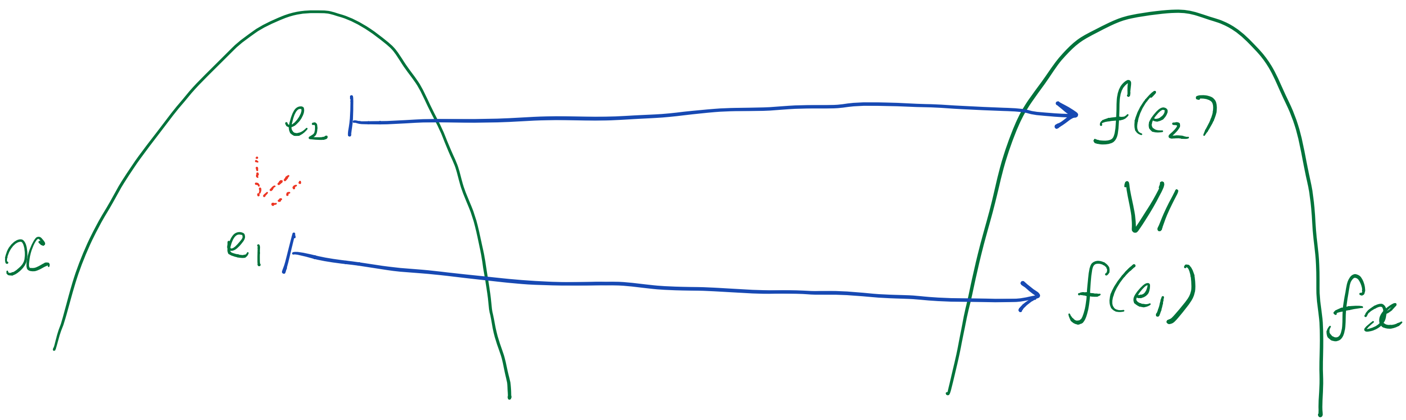
Maps preserve concurrency, and locally reflect causal dependency:

$$\forall x \in \mathcal{C}(E), e_1, e_2 \in x.\ f(e_1) \leqslant f(e_2) \implies e_1 \leqslant e_2\,.$$



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$\rightsquigarrow$

Semantics of synchronising processes [Hoare, Milner] can be expressed in terms of universal constructions on event structures;
in games, pullbacks and partial-total factorisation play a central role.

Relations between models via adjunctions.
E.g. the event-structure unfolding of a basic Petri net is a right adjoint.
Coreflection of event structures in stable families v useful for constructions.

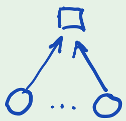
Strong bisimulation via open maps, defined diagrammatically.
$\rightsquigarrow$ Preheaf models for concurrency ...

Symmetry as "self bisimulation" helps compensate for the overly-concrete nature of models for concurrency. *E.g.*, unfolding of Petri nets with multiplicities defined universally only up to symmetry.

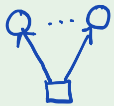
Remark: Petri nets as containers (Thanks Fredrik!)

A basic Petri net with events E and conditions B can be seen as a pair of containers, one associating

events with their preconditions: $(E \rhd Pre)$ where $Pre : E \to \mathcal{P}(B)$

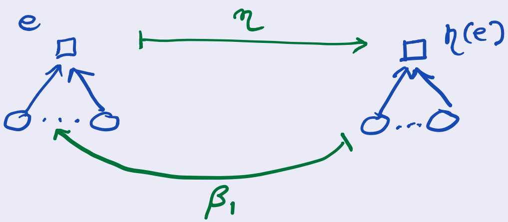


events with their postconditions: $(E \rhd Post)$ where $Post : E \to \mathcal{P}(B)$



(Its initial marking identified with the postconds of a distinguished initial event)

A (total) map $(\eta, \beta) : N \to N'$ of basic Petri nets can be reformulated as a pair of container maps :



$(\eta, \beta_1) : (E \rhd Pre) \to (E' \rhd Pre')$

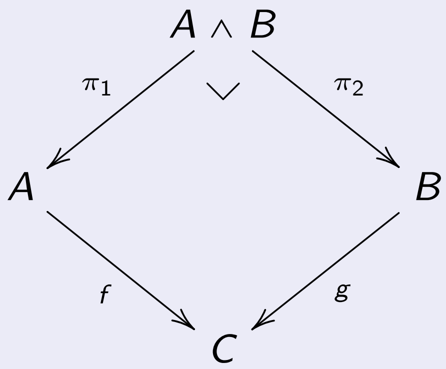
$(\eta, \beta_2) : (E \rhd Post) \to (E' \rhd Post')$

(with $\eta : E \to E'$ preserving initial events, ...)

Quantum Petri nets are being used to formalise quantum strategies.

Pullbacks - for composing processes with a common interface

Total maps $f : A \to C$ and $g : B \to C$ have pullbacks in the category of event structures:



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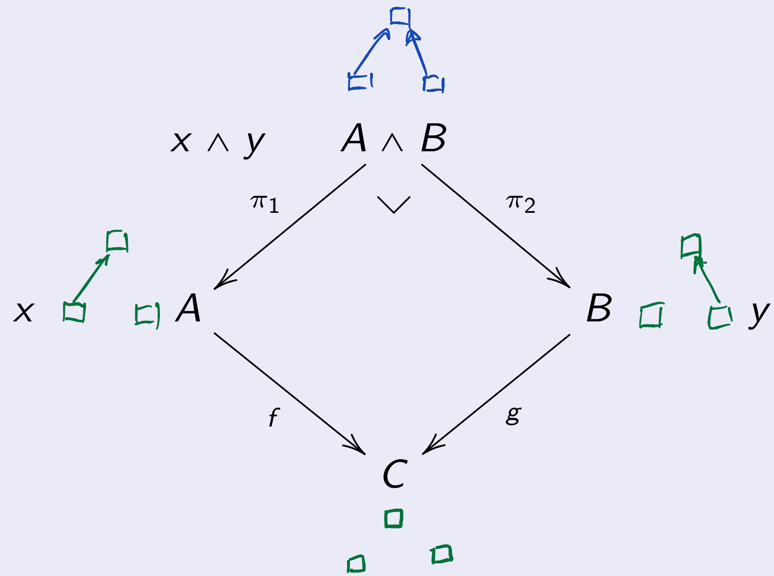


Finite configurations of $A \wedge B$ correspond to the composite bijections

$$x \wedge y : x \cong fx = gy \cong y$$

between configurations $x \in \mathcal{C}(A)$ and $y \in \mathcal{C}(B)$ s.t. $fx = gy$ which are secured bijections, *i.e.* for which the transitive relation generated on $x \wedge y$ by

$$(a, b) \leqslant (a', b') \text{ if } a \leqslant_A a' \text{ or } b \leqslant_B b'$$

is a partial order.

Defined part of a map - for hiding

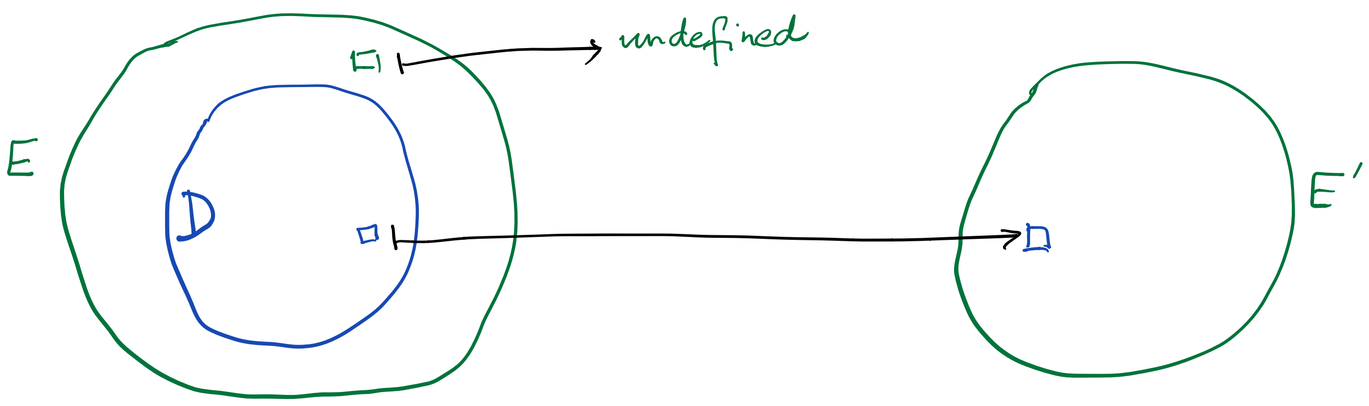
A partial map

$$f : E \rightarrow E'$$

of event structures has partial-total factorization as a composition

$$E \xrightarrow{p} D \xrightarrow{t} E'$$

where $t : D \rightarrow E'$ is the defined part of f.



Games, a paradigm for interaction [Conway, Joyal]

The dichotomy Player vs. Opponent has many readings:
Team of Players vs. Team of Opponents; Allies vs. Enemies;
Prover vs. Disprover; Process vs. Environment

Operations on (2-party) games:

Dual game $G^{\perp}$ *- interchange the role of Player and Opponent;*
Counter-strategy = strategy for Opponent = strategy for Player in dual game.

Parallel composition of games $G \| H$.

A strategy (for Player) from a game G *to a game* H *is a strategy in* $G^{\perp} \| H$.
A strategy (for Player) from a game H *to a game* K *is a strategy in* $H^{\perp} \| K$.

Compose by letting them play against each other in the common game H.

The Copycat strategy in $G^{\perp} \| G$, *so from* G *to* G ...

Concurrent games

Games and strategies are represented by event structures with polarity, an event structure where events carry a polarity ⊞ / ⊟ *(Player/Opponent).*

Maps are those of event structures which preserve polarity.

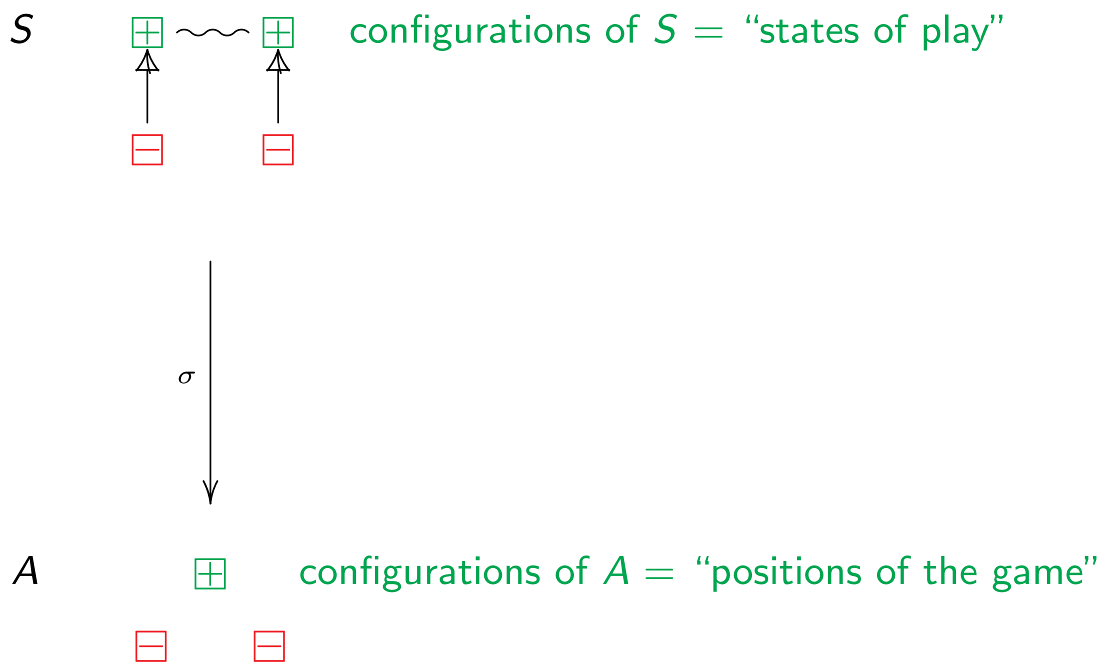
Dual, $B^\perp$, of an event structure with polarity B is a copy of the event structure B with a reversal of polarities; this switches the roles of Player and Opponent.

(Simple) Parallel composition: $A \| B$, by consistent juxtaposition.

A strategy from a game A to a game B is a strategy in $A^\perp \| B$, written

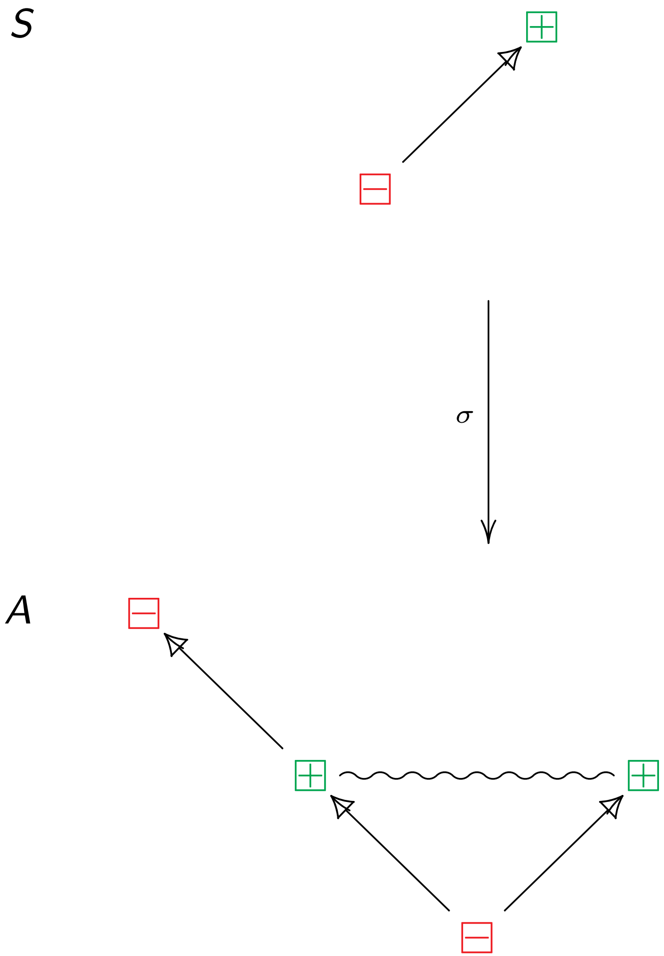
$$\sigma : A \longrightarrow\!\!\!\!\!\!+\!\!\!\!\!> B$$

A strategy in a game A is a special total map $\sigma : S \to A$, *e.g.*



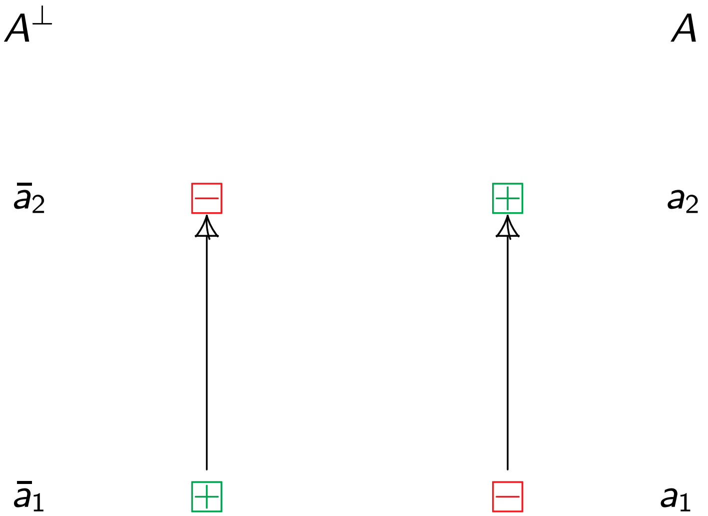
The strategy: answer either move of Opponent by the Player move.

When games are trees



The strategy: force Opponent to get stuck.

Copycat strategy from A to A



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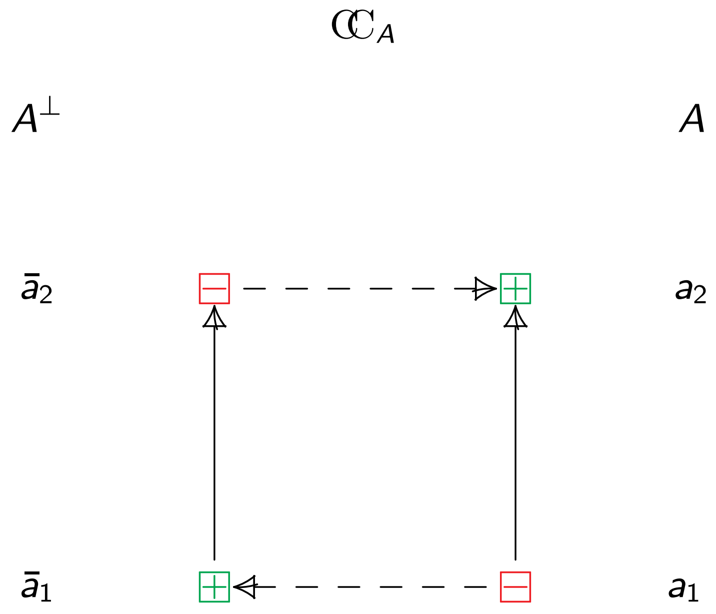
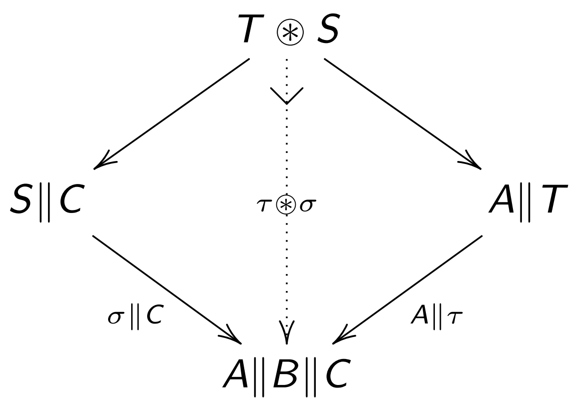


Composition of strategies $\sigma : A \mathrel{+\!\!\!\!\rightarrow} B$ and $\tau : B \mathrel{+\!\!\!\!\rightarrow} C$

To compose

synchronise complementary moves over common game B via pullback:



Composition of strategies $\sigma : A \nrightarrow B$ and $\tau : B \nrightarrow C$

To compose

$$\begin{array}{ccc} S & & T \\ \sigma \downarrow & & \downarrow \tau \\ A^{\perp} \| B & & B^{\perp} \| C \end{array}$$

synchronise complementary moves over common game B via pullback; then hide synchronisations via partial-total factorisation:

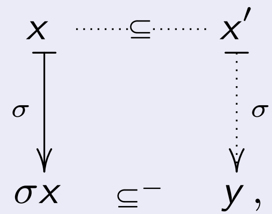
$$\begin{array}{ccccc} \textit{before hiding} \quad T \circledast S & \cdots\cdots\rightarrow & T \odot S \quad \textit{after hiding} \\ \tau \circledast \sigma \downarrow & & \downarrow \tau \odot \sigma \\ A^{\perp} \| B \| C & \cdots\cdots\rightarrow & A^{\perp} \| C \end{array}$$

Conditions on a strategy are those needed to make copycat identity w.r.t. composition.

For copycat to be identity w.r.t. composition

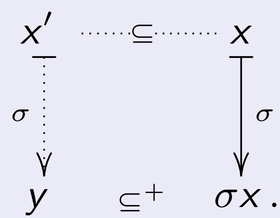
a strategy in a game A has to be $\sigma : S \to A$, a total map of event structures with polarity, which is

(i) whenever $\sigma x \subseteq^- y$ *in* $\mathcal{C}(A)$ *there is a unique* $x' \in \mathcal{C}(S)$ *s.t.* $x \subseteq x'$ & $\sigma x' = y$, *i.e.*



A strategy should be receptive to Opponent moves allowed by the game.

(ii) whenever $y \subseteq^+ \sigma x$ *in* $\mathcal{C}(A)$ *there is a (necessarily unique)* $x' \in \mathcal{C}(S)$ *s.t.* $x' \subseteq x$ & $\sigma x' = y$, *i.e.*



A strategy should only adjoin immediate causal dependencies $\boxminus \rightarrow \boxplus$.

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$$\begin{array}{ccc} x & \cdots\subseteq\cdots & x' \\ \sigma \downarrow & & \vdots\, \sigma \\ \sigma x & \subseteq^{-} & y\,, \end{array}$$

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$\rightsquigarrow$ compact-closed bicategory of concurrent games and strategies.

Strategies as profunctors

Defining the Scott order on configurations of A

$$y \sqsubseteq_A x \;\text{ iff }\; y \supseteq^- \cdot \subseteq^+ \cdot \supseteq^- \cdots \supseteq^- \cdot \subseteq^+ x$$

we obtain a partial order and a factorization system:

$$\begin{array}{cccc} & & & x \\ & & \sqsubseteq\!\!\!\!\nearrow & \cup\!\!\!\mid^{+} \\ \exists! z. & y & \supseteq^- & z\,. \end{array}$$

Proposition $z \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ *iff* $z_2 \sqsubseteq_A z_1$.

Theorem *Strategies* σ *correspond to discrete fibrations, i.e.,*

$$\begin{array}{cccc} \exists! x'. & x' & \cdots\sqsubseteq_S\cdots & x \\ & \sigma``\downarrow & & \downarrow\sigma`` \\ & y & \sqsubseteq_A & \sigma x\,, \end{array}$$

which preserve $\supseteq^-$, $\subseteq^+$ *and* $\varnothing$. *So strategies* $\sigma : A \,\mathrlap{\,+}{\longrightarrow}\, B$ *correspond to (certain) profunctors* $\sigma`` : (\mathcal{C}(A), \sqsubseteq_A) \,\mathrlap{\,+}{\longrightarrow}\, (\mathcal{C}(B), \sqsubseteq_B)$.

$\rightsquigarrow$ *Lax functors from strategies to profunctors, and to Scott domains ...*

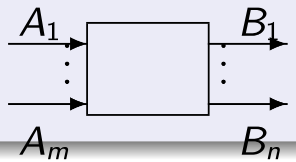
A language for concurrent strategies

Types: *Games A, B, C, ... with operations $A^{\perp}$, $A \| B$, sums $\sum_{i \in I} A_i$, recursively-defined types, ...*

A **term**

$$x_1 : A_1, \cdots, x_m : A_m \vdash \; t \; \dashv y_1 : B_1, \cdots, y_n : B_n \;,$$

denotes a strategy from $A_1 \| \cdots \| A_m$ to $B_1 \| \cdots \| B_n$.



Idea: t denotes a strategy $S \to \vec{A}^{\perp} \| \vec{B}$.
The term t describes witnesses, finite configurations of S, to a relation between finite configurations $\vec{x}$ of $\vec{A}$ and $\vec{y}$ of $\vec{B}$. Cf. profunctors.

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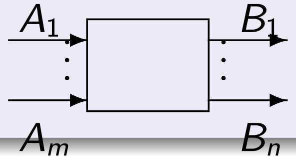

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Copycat $x : A \vdash \; y \sqsubseteq_A x \; \dashv y : A$ *and other terms "wiring in" causality.*

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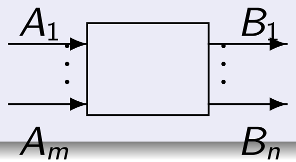

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Composition $\dfrac{\Gamma \vdash t \dashv \Delta \qquad \Delta \vdash u \dashv \mathrm{H}}{\Gamma \vdash \exists \Delta. [\, t \parallel u \,] \dashv \mathrm{H}}$

Duality $\dfrac{A, \Gamma \vdash t \dashv \Delta}{\Gamma \vdash t \dashv A^\perp, \Delta}$ (double line)

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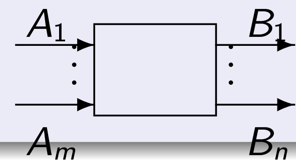

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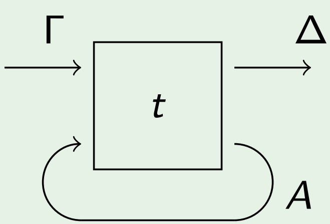
Feedback



$$\Gamma \vdash \exists x : A, y : A^{\perp} . [\, x \sqsubseteq_A y \parallel t \,] \dashv \Delta$$

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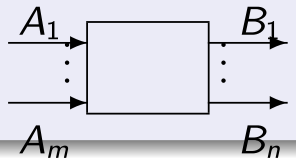

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Duality $\dfrac{A, \Gamma \vdash t \dashv \Delta}{\Gamma \vdash t \dashv A^\perp, \Delta}$

Sum $\Sigma_{i\in I} t_i$

Conjunction $t_1 \wedge t_2$

Special cases - recovering functions 1

A concurrent strategy is deterministic when conflicting behaviour of Player implies conflicting behaviour of Opponent.

Stable spans and stable functions The sub-bicategory where the events of games are purely +ve is that of **stable spans** used in nd dataflow; feedback given by trace.

$$S \xrightarrow{\sigma} A^{\perp} \| B$$

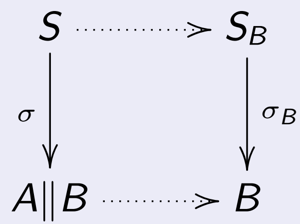
Its deterministic sub-bicategory **Stable** is equivalent to **stable functions between Berry domains** (coherent w.r.t. countable event structures with binary conflict); Girard's **coherence spaces** when causal dependency trivial.

Open games?

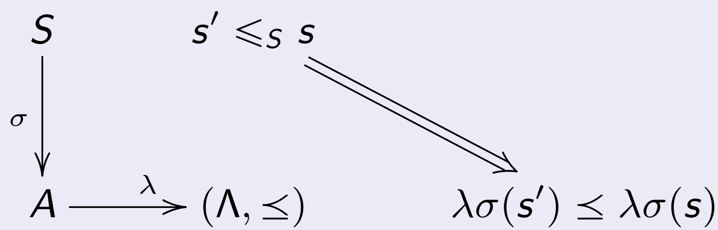
Special cases - recovering functions 2

Two tools for recovering functions in **parts** of a game:

1. Projecting a strategy to a parallel component of a game yields a strategy:



2. *Imperfect information* via an "access order" $(\Lambda, \preceq)$ on moves of the game; causal dependency of the game and additional causal dependencies of the strategy must respect it:



Open games via a dialectica category (thanks Jules!), *e.g.*

The dialectical category with maps

$(f, g) : \begin{pmatrix} X \\ R \end{pmatrix} \to \begin{pmatrix} Y \\ S \end{pmatrix}$ where $f : X \to Y$ and $g : X \times S \to R$ in **Stable**

embeds fully and faithfully in the sub-bicategory of strategies comprising deterministic strategies in games

$$(X^+ \| R^-) \twoheadrightarrow (Y^+ \| S^-) = (X^+ \| R^-)^\perp \| (Y^+ \| S^-) \text{ with access order}$$

$$\begin{array}{ccc} X^- & \prec & Y^+ \\ & & \curlywedge \\ R^+ & \succ & S^- \end{array}$$

Their deterministic counterstrategies correspond to configurations of X paired with $h : Y \to S$ in **Stable**:

$$\begin{array}{ccc} X^+ & \prec & Y^- \\ & & \curlywedge \\ R^- & \succ & S^+ \end{array}$$

Now have all the ingredients for open games w.r.t. **Stable** (and **Stable spans**).