Quantitative containers

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joint work with Georgi Nakov

Linear logic

In "ordinary" logic:

$$x : A, f : A \to B \vdash (x, fx) : A \times B$$

In the "real" world:

$$x : \mathrm{Apple}, \mathrm{oven} : \mathrm{Apple} \multimap \mathrm{Pie} \nvdash \mathrm{Apple} \otimes \mathrm{Pie}$$

Captured by Linear logic [Girard 1987].

In Computer Science, useful for I/O, communication channels, memory management, ...

Dependent type theory

A foundation for constructive mathematics [Martin-Löf 1972].

A functional programming language [Martin-Löf 1982].

Key point: type system expressive enough to encode logical propositions, e.g.

$$\text{head} : \text{List } A\ (n+1) \to A$$

$$\text{sort} : \text{List } A\ n \to \text{SortedList } A\ n$$

Good for functional correctness, but what about resource safety etc?

Combining linear and dependent types

A difficult problem!

Is

$$\mathrm{refl} : (x : A) \to x =_A x$$

linear?

Is

$$\mathrm{divide} : (n : \mathbb{N}) \to (m : \mathbb{N}) \to m > 0 \to \mathbb{N}$$

linear?

Is

$$\mathrm{sort} : (xs : \mathrm{List}\ A\ n) \to \left((xs' : \mathrm{SortedList}\ A\ n) \times (\mathrm{Permutation}\ xs\ (U(xs')))\right)$$

linear?

Linear dependent types

Early attempts generalizing linear/non-linear logic [Krishnaswami et al 2015, Vákár 2015], splitting context into a linear and a non-linear region:

$$\Gamma ; \Delta \vdash a : A$$

Importantly, types may only depend on non-linear terms, i.e. are only formed when $\Delta = \emptyset$.

$\Rightarrow$ Question how to count occurrences in types goes away. ✓

$\Rightarrow$ But, cannot prove any properties of linear terms. ?

Quantitative Type Theory

Core idea: It is still possible to contemplate consumed things. [McBride 2016]

Rather than erasing things from the context, we record their usage, e.g.

$$y \overset{0}{:} \text{Cheese}, x \overset{2}{:} \text{Apple}, \text{oven} \overset{1}{:} \text{Apple} \multimap \text{Pie} \vdash (x, \text{oven}(x)) : \text{Apple} \otimes \text{Pie}$$

In general: annotations from a semiring $(R, +, \cdot, 0, 1)$.

$$\frac{\Gamma \vdash f : (x \overset{\rho}{:} A) \to B[x] \qquad \Gamma' \vdash a : A}{\Gamma + \rho\Gamma' \vdash f\,a : B[a]}$$

Importantly: forming types do not consume resources, i.e. it happens in contexts of the form $0 \cdot \Gamma$.

That is: occurances in types are free, and we can still prove properties of linear things.

Categorical semantics [Atkey 2018]

Quantitative extension of Categories with families [Dybjer 1995].

$\mathbb{C}$ category — contexts and substitutions

$\mathrm{Ty} : \mathbb{C}^{op} \to \mathrm{Set}$ — types and subst. actions

$\mathrm{Tm} : (\Gamma \in \mathbb{C}^{op}) \to \mathrm{Ty}(\Gamma) \to \mathrm{Set}$ — terms and — " —

$\_ \bullet \_ : (\Gamma \in \mathbb{C}) \to \mathrm{Ty}(\Gamma) \to \mathbb{C}$ — context extension (with universal property)

In addition:

$\mathbb{L}$ category — resourced contexts and subs

$U : \mathbb{L} \to \mathbb{C}$ — underlying context

$\rho(-) : \mathbb{L} \to \mathbb{L}$ for each $\rho \in R$ — scaling

$(+) : \mathbb{L} \times_U \mathbb{L} \to \mathbb{L}$ — context addition

$\mathrm{RTm} : (\Gamma \in \mathbb{L}^{op}) \to \mathrm{Ty}(U\Gamma) \to \mathrm{Set}$ — resourced terms (over Tm)

$\_ \cdot \rho \, \_ : (\Gamma \in \mathbb{L}) \to \mathrm{Ty}(U\Gamma) \to \mathbb{L}$ for each $\rho \in R$

resourced context extension (over $\_ \bullet \_$)

Concrete models

SKI

BCI

1. Take $\mathbb{C}$ any CwF, $\mathbb{L} = \mathbb{C}$, $U = id$

"programs" "application" $(\cdot) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$

"duplication" $!_p : \mathcal{A} \to \mathcal{A}$

2. Fix an R-linear combinatory algebra $(\mathcal{A}, (\cdot), !_p, \underbrace{B, C, I, K, W, D, \delta, F}_{\text{combinators, e.g. } B \cdot x \cdot y \cdot z = x \cdot (y \cdot z)})$.

$K \cdot x \cdot !_0 y = x$

An assembly $X = (|X|, \Vdash_X)$ is a set $|X|$ and a relation $\Vdash_X \subseteq \mathcal{A} \times |X|$.

A morphism $X \to Y$ is a function $f : |X| \to |Y|$ s.t. there exists $a_f \in \mathcal{A}$ realising f: if $a \Vdash_X x$ then $a_f \cdot a \Vdash_Y f(x)$.

graphs of linear functions on $\mathbb{N}$

Take $\mathbb{L} = Asm(\mathcal{A})$, $\mathbb{C} = Set$, $U = |-|$. Concretely can take $\mathcal{A} = P(\omega)_{lin}$ [Hoshino 2007].

3. Relational realisability models: $\mathbb{L} = ReflGraph(Asm(\mathcal{A}))$, $\mathbb{C} = ReflGraph(Set)$.

Some type formers

- $(x \overset{\rho}{:} A) \to P[x]$ dep. functions using argument ρ times
- $(x \overset{\rho}{:} A) \otimes P[x]$ dep pairs with ρ copies of first component
- I monoidal unit (type)
- $\top$ terminal type
- $A \oplus B$ additive disjunction (coproduct)
- $A \,\&\, B$ additive conjunction ("pick one — your choice")
- ⋮

$$!_\rho A := (x \overset{\rho}{:} A) \otimes I$$

$$[\![I]\!] = (\{*\},\ x \Vdash_I * \iff x = \{I\})$$

$$[\![\top]\!] = (\{*\},\ x \Vdash * \iff \text{true})$$

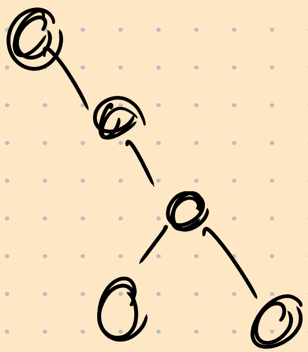
Data types?

How to add the trees, lists, natural numbers etc that we all know and love?

If done ad-hoc, how do we know elimination principle is right?

Instead, let's take a principled approach and consider initial algebras of containers.

Containers 101 [Abbott et al 2003]



A container is given by $S : Set$ "shapes"

$P : S \to Set$ "positions"

Represents functor $F : Set \to Set$

$$F(X) = (s : S) \times (P(s) \to X)$$

Algebra for F is a pair (X, c) where $X : Set$ "carrier"

$c : F(X) \to X$ "constructor"

Morphism $(X, c) \to (X', c')$ is $f : X \to X'$ s.t.

$$\begin{tikzcd} FX \arrow[r, "c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "G"] & X \arrow[d, "f"] \\ FX' \arrow[r, "c'"'] & X' \end{tikzcd}$$

An algebra (X, c) is initial if there is a unique algebra morphism $(X, c) \to (Y, d)$ for every algebra (Y, d)

Quantitative containers

$$F(X) = (s \overset{?}{:} S) \otimes (P(s) \multimap X)$$

Functor on cat. of ~~closed~~ types and linear functions:

over fixed context $\Delta = 0\Delta$

Objects T s.t. $\Delta \vdash T$ type

Morphisms (Γ, f) s.t. $\Gamma \vdash f : T \multimap T'$, $0\Gamma = \Delta$

$id = (\Delta, \lambda x.x)$

$(\Gamma, f) \circ (\Gamma', f') = (\Gamma + \Gamma', f \circ f')$

$$f : X \multimap Y \implies F(f) : F(X) \multimap F(Y)$$

$$F(\Gamma, f) = (\Gamma, \dots)$$

Hence we can consider category of F-algebras

Initial F-algebras

Can construct initial algebras for finitary containers in $\mathrm{Asm}(P(\omega))$:

- Underlying set constructed using initial algebras in metatheory.

  E.g. $F(X) = I \oplus X \quad |\mu F| = \mathbb{N}$

- By induction on elements in metatheory, define realisers $r(x) \in \omega$.

  E.g. $r(0) = 0 \qquad r(n+1) = \langle 1, r(n) \rangle$

  Define $x \Vdash_{\mu F} t \iff x = \{r(t)\}$.

- Check that constructors and mediating map $\mu F \to X$ are realised, again by meta-induction.

Induction

$$F(X) = (s \overset{1}{:} S) \otimes (P(s) \multimap X)$$
$$c : F(W) \to W$$

What about the elimination principle?

$$\frac{Q : W \to \mathrm{Type} \qquad M : (s \overset{1}{:} S)(h \overset{0}{:} P(s) \multimap W)(ih \overset{1}{:} (y : P(s)) \to Q(h\,y)) \to Q(c(s,h))}{\mathrm{elim}(Q, M) : (x \overset{1}{:} W) \to Q[x]}$$

Folklore construction [Hermida & Jacobs 1998] to derive it from initiality in traditional setting:

- Use c, M to make $(y \overset{0}{:} W) \overset{\&?}{\otimes} Q[y]$ into an F-algebra; hence get $\mathrm{fold} : W \to (y \overset{0}{:} W) \otimes Q[y]$
- Compose with $\mathrm{snd} : (p : (y : W) \times Q[y]) \to Q[\mathrm{fst}\,p]$ to get $(x : W) \to Q[\overbrace{\mathrm{fst}(\mathrm{fold}}^{\mathrm{id}}\, x)]$
- Prove that $\mathrm{fst} : (y : W) \times Q[y] \to W$ is F-algebra morphism; hence so is $\mathrm{fst} \circ \mathrm{fold} : W \to W$ and by uniqueness $\mathrm{fst} \circ \mathrm{fold} = \mathrm{id}$. Hence we have $(x : W) \to Q[x]$ as required.

$$\begin{tikzcd}
FW \arrow[r, "c"] \arrow[d] & W \arrow[d, "\mathrm{fold}"] \arrow[dd, bend left, "\mathrm{id}"] \\
F(\ldots) \arrow[r] \arrow[d] & (y \overset{0}{:} W) \times Q \arrow[d, "\mathrm{fst}"] \\
FW \arrow[r, "c"'] & W
\end{tikzcd}$$

The lack of normality

Polynomial functors traditionally inductively generated by

$$\mathrm{Id} \mid \mathrm{Const}_A \mid (+) \mid (\times) \mid A \to -$$

Thm [Abbot et al 2005] Every such polynomial functor has a container normal form $F(X) \cong (s : S_F) \times (P_F(s) \to X)$

E.g. $F(X) = 1 + X \times X \cong (b : 2) \times \left(\begin{pmatrix} \text{if } b \text{ then } 0 \\ \text{else } 2 \end{pmatrix} \to X \right)$

This breaks down in QTT setting.

E.g. $(0 \multimap X) \cong \top \not\cong I$

$(2 \multimap X) \cong X \,\&\, X \not\cong X \otimes X$

We can do it by hand

Instead of computing the CNF and deriving its induction principle, we can formulate it directly.

Main step is to inductively compute the predicate lifting $\hat{F} : (Q : X \to \mathrm{Type}) \to (FX \to \mathrm{Type})$, which encodes the I.H. for the elim. principle.

available, since we are contemplating a type

E.g. $$\widehat{F \otimes G}(Q, z) = \hat{F}(Q, \mathrm{fst}\, z) \otimes \hat{G}(Q, \mathrm{snd}\, z)$$

Derivation of elim. from initiality works the same, mutatis mutandis.

Summary and outlook

QTT combining linear and dependent types

Initial algebras of polynomial functors as principled data types for QTT

Constructing μF for non-finitary F in concrete models?

Dependent resources to allow $F(X) = (s \overset{1}{:} S) \otimes !_{p(s)} (P(s) \multimap X)$?

External "semantic" description of quantitative polynomial functors?

Extending permutation-preservation [Atkey and Wood 2018] to arbitrary containers?

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