## Moens' theorem and fibered toposes

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# Plan of talk

- Elementary toposes and Grothendieck toposes
- Realizability toposes
- Fibered categories
- Characterizing realizability toposes

## Elementary toposes and Grothendieck toposes

### Elementary toposes

Definition (Lawvere, ca. 1970)

### An elementary topos is a category $\mathcal{E}$ with

- finite limits
- exponential objects  $B^A$  for  $A, B \in \mathcal{E}$  (cartesian closed)
- a subobject classifier, i.e. a morphism t : 1 → Ω such that for every monomorphism m : U → A there exists χ : A → Ω making

$$U \longrightarrow 1$$

$$\downarrow^{m} \downarrow \qquad \qquad \downarrow^{t}$$

$$A \longrightarrow \Omega$$

a pullback.

# Grothendieck toposes

#### Grothendieck toposes

Grothendieck toposes can equivalently be defined in the following ways:

- Introduced around 1960 by G. as categories of sheaves on a site
- Characterized 1963 by Giraud as locally small ~-pretoposes with a separating set of objects
- ④ Equivalently: elementary topos *E* admitting a (necessarily unique) bounded geometric morphism *E* → Set
- Inspired by 3, define a Grothendieck topos over an (elementary) base topos S as a bounded geometric morphism  $\mathcal{E} \to S$

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What do all these words mean??

## Locally small, separating set

- C is called locally small, if the 'homsets' C(A, B) are really sets, as opposed to proper classes
- A separating set of objects in  $\mathbb{C}$  is a family  $(C_i)_{i \in I}$  of objects indexed by a set I such that for all parallel pairs  $f, g : A \to B$  we have

 $(\forall i \in I \forall h : C_i \rightarrow A. fh = gh) \Rightarrow f = g.$ 



Regular categories

### $\infty$ -pretopos = exact $\infty$ -extensive category

= effective regular  $\infty$ -extensive category

#### Definition

A **regular category** is a category with finite limits and pullback-stable regular-epi/mono factorizations.



# $\infty$ -Pretoposes

Exact categories

• An *equivalence relation* in a f.l. category  $\mathbb{C}$  is a jointly monic pair  $r_1, r_2 : R \to A$  such that for all  $X \in \mathbb{C}$ , the set

 $\{(r_1x,r_2x) \mid x:X \to R\}$ 

is an equivalence relation on  $\mathbb{C}(X, A)$ 

• The kernel pair of any morphism  $f : A \rightarrow B$  – given by the pullback

$$X \longrightarrow A$$

$$r_{2} \downarrow \Box^{r_{1}} \downarrow \downarrow^{r_{1}} \downarrow^{r_{1}} \downarrow^{r_{1}}$$

$$A \xrightarrow{f} B$$

is always an equivalence relation

#### Definition

An **exact** (or **effective regular**) category is a regular category in which every equivalence relation is a kernel pair.

### $\infty$ -Pretoposes

Extensive categories

Assume  $\mathbb{C}$  has finite limits and small coproducts

• Coproducts in C are called **disjoint**, if the squares

$$\begin{array}{lll} 0 \longrightarrow A_i & & A_i \longrightarrow A_i \\ \psi & \psi & (i \neq j) & \text{and} & \psi & \psi \\ A_j \Rightarrow \coprod_{i \in I} A_i & & A_i \Rightarrow \coprod_{i \in I} A_i \end{array}$$

are always pullbacks

• Coproducts in  $\mathbb{C}$  are called **stable**, if for any  $f: B \to \coprod_{i \in I} A_i$ , the family

$$(B_i \xrightarrow{\sigma_i} B)_{i \in I}$$
 given by pullbacks

$$\begin{array}{c} B_{i} \xrightarrow{\sigma_{i}} & B \\ \downarrow^{-} & \downarrow^{f} \\ A_{i} \xrightarrow{} & \coprod_{i \in I} A_{i} \end{array}$$

represents B as coproduct of the  $B_i$ 

#### Definition

An  $\infty$ -(I)extensive category is a category  $\mathbb{C}$  with finite limits and disjoint and stable small coproducts.

### $\infty$ -Pretoposes

#### Examples

- Complete lattices (A, ≤) viewed as categories have finite limits and small coproducts, but these are not disjoint – coproducts are stable precisely for *complete Heyting algebras*
- Top (topological spaces) and Cat (small categories) are ∞-extensive but not regular
- Monadic categories over Set are always exact and have small coproducts, but are rarely extensive

#### Definition

An  $\infty$ -pretopos is a category which is exact and  $\infty$ -extensive.

#### Examples

- Grothendieck toposes
- the category of *small* presheaves on Set

### Geometric morphisms

• A geometric morphism  $\mathcal{E}\to \mathcal{S}$  between toposes  $\mathcal{E}$  and  $\mathcal{S}$  is an adjunction

 $(\Delta: S \to \mathcal{E}) \dashv (\Gamma: \mathcal{E} \to S)$ 

of f.l.p. functors ( $\Delta$  is the 'inverse image part';  $\Gamma$  the 'direct image part')

- $(\Delta \dashv \Gamma)$  is called **bounded**, if there exists  $B \in \mathcal{E}$  such that for every  $E \in \mathcal{E}$  there exists a subquotient span  $B \times \Delta(S) \leftarrow \bullet \twoheadrightarrow E$
- It is called localic if it is bounded by 1
- If  $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \textbf{Set}$ , then we necessarily have

$$\Delta(J) = \sum_{j \in J} 1$$
 and  $\Gamma(A) = \mathcal{E}(1, A)$ 

for  $J \in \mathbf{Set}$  and  $A \in \mathcal{E}$ 

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#### Remark

Without the bound in 3,  $\mathcal{E}$  need not be cocomplete. Example: subcategory of  $\widehat{\mathbb{Z}}$  on actions with uniform bound on the size of orbits.

Realizability toposes

## Realizability toposes

- Were introduced in 1980 by Hyland, Johnstone, and Pitts
- Not Grothendieck toposes
- Most well known: Hyland's effective topos *Eff* 'Universe of constructive recursive mathematics'
- usually constructed via triposes

## Partial combinatory algebras

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Definition
A PCA is a set \mathcal{A} with a partial binary operation
                                                (-\cdot -): \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}
having elements k, s \in \mathcal{A} such that
                  (i) k \cdot x \cdot y = x (ii) s \cdot x \cdot y \downarrow (iii) s \cdot x \cdot y \cdot z \preceq x \cdot z \cdot (y \cdot z)
for all x, y, z \in \mathcal{A}.
Example
First Kleene algebra: (\mathbb{N}, \cdot) with
                                        n \cdot m \simeq \phi_n(m) for n, m \in \mathbb{N},
```

where  $(\phi_n)_{n \in \mathbb{N}}$  is an effective enumeration of partial recursive functions.

## Fibrations from PCAs

PCA  $\mathcal{A}$  gives rise to indexed preorders  $fam(\mathcal{A}), rt(\mathcal{A}) : \textbf{Set}^{op} \to \textbf{Ord}.$ 

• Family fibration:  $fam(\mathcal{A})(J) = (\mathcal{A}^J, \leq)$ , with

 $\varphi \leq \psi \quad :\Leftrightarrow \quad \exists e \in \mathcal{A} \; \forall j \in J \, . \, e \cdot \varphi(j) = \psi(i)$ 

for  $\varphi, \psi: J \to \mathcal{A}$ .

• Realizability tripos:  $rt(A)(J) = ((PA)^J, \leq)$ , with

 $\varphi \leq \psi \quad :\Leftrightarrow \quad \exists e \in \mathcal{A} \; \forall j \in J \; \forall a \in \varphi(j) . \; e \cdot a \in \psi(i)$ 

for  $\varphi, \psi : J \to PA$ .

#### **Observations**

- fam(A) has indexed finite meets
- rt(A) models full 1st order logic
- both have generic predicates
- rt(A) is free cocompletion of fam(A) under  $\exists$  (Hofstra 2006)

# Realizability toposes

#### Definition

- The **realizability topos RT**(*A*) over *A* is the category of partial equivalence relations and compatible functional relations in *A* (details omitted)
- The constant objects functor Δ : Set → RT(A) maps J ∈ Set to (J, δ<sub>J</sub>) (discrete/diagonal equivalence relation)
- **RT**(*A*) is never a Grothendieck topos (except for the trivial pca)
- ▲ is bounded by 1, but not the inverse image part of a geometric morphism
- it makes sense to compare constant objects functors and inverse image functors, since both are instances of the same construction in the context of triposes

Fibered Categories

# $\Delta$ and gluing fibrations

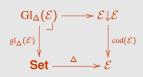
Goal: Understand inverse image functors

```
(\Delta: \mathbf{Set} \to \mathcal{E}) \dashv \Gamma
```

and constant objects functors

 $\Delta$  : **Set**  $\rightarrow$  **RT**( $\mathcal{A}$ )

better by looking at their gluing fibrations, defined by the pullback



# Fibered category theory

#### References

- Jean Bénabou, Fibered categories and the foundations of naive category theory, 1985
- Thomas Streicher, *Fibred categories à la Jean Bénabou*, unpublished, 1999-2012
- Peter Johnstone, Sketches of an Elephant, 2003

#### Idea/Philosophy

- *Elementary category theory*: finitary conditions, first order axiomatizable, no size conditions, avoid ZFC (f.l. category, elementary topos)
- Naive category theory: not concerned about formal, foundational aspects, use size conditions and make reference to Set freely
- Bénabou proposes fibrations to reconcile both, fibrations allow to express 'non-finitary conditions' in an elementary manner
- generalize and form analogies from family fibrations

# Family fibrations

#### Definition

Let  $\mathbb{C}$  be a category.

The category Fam(ℂ) has families (C<sub>i</sub>)<sub>i∈I</sub> of objects of ℂ as objects; a morphism (C<sub>i</sub>)<sub>i∈I</sub> → (D<sub>j</sub>)<sub>j∈J</sub> is a pair

 $(u: I \rightarrow J, (f_i: C_i \rightarrow D_{ui})_{i \in I}.$ 

• The family fibration of C is the functor

$$\begin{array}{rcl} \operatorname{fam}(\mathbb{C}) & : & \operatorname{Fam}(\mathbb{C}) & \to & \operatorname{\mathbf{Set}} \\ & & (C_i)_{i \in I} & \mapsto & I \\ & & (u, (f_i)_{i \in I}) & \mapsto & u \end{array}$$

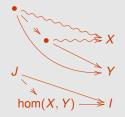
mapping  $(C_i)_{i \in I} \operatorname{fam}(\mathbb{C}) : \operatorname{Fam}(\mathbb{C}) \to \operatorname{Set}$  of a category  $\mathbb{C}$  is the fibration having

### Local smallness

#### Definition

Let  $P : \mathbb{X} \to \mathbb{B}$  be a fibration,  $I \in \mathbb{B}$ ,  $X, Y \in P(I)$ . A family of morphisms

from X to Y is a span  $X \leftarrow c \to f \to Y$  where P(c) = P(f) and c is cartesian. P is called **locally small**, if for every pair  $X, Y \in P(I)$  there exists a *universal* family of morphisms (terminal among such spans).



#### Lemma

A category  $\mathbb{C}$  is locally small, iff fam( $\mathbb{C}$ ) is locally small in the above sense.

## Finite limit fibrations

... towards extensive fibratiions and Moens' theorem

#### Definition

Let  $\mathbb{B}$  be a f.l. category. A **finite limit fibration** on  $\mathbb{B}$  is a fibration  $P : \mathbb{X} \to \mathbb{B}$  satisfying either of the following equivalent definitions.

- X has finite limits and P preserves them
- All fibers P(I) have finite limits, and they are preserved under reindexing

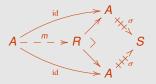
#### Lemma

A category  $\mathbb{C}$  has finite limits iff fam( $\mathbb{C}$ ) is a finite limit fibration.

## Extensive fibrations

Let  $P : \mathbb{X} \to \mathbb{C}$  be a finite limit fibration.

- *P* is said to have internal sums, if it is also an opfibration
   (*P*<sup>op</sup> : X<sup>op</sup> → C<sup>op</sup> is a fibration), and cocartesian maps in X are stable
   under pullback along cartesian maps ('Beck-Chevalley condition')
- *P* is said to have **stable internal sums**, if cocartesian maps are stable under pullback along *arbitrary* maps in X
- Internal sums are called **disjoint**, if the mediating arrow *m* in the diagram



is cocartesian for every cocartesian map  $\sigma: A \rightarrow S$  in X

• An extensive fibration is a finite-limit fibration with stable disjoint internal sums.

#### Lemma

```
A category \mathbb{C} is \infty-extensive iff fam(\mathbb{C}) is extensive.
```

### Moens' theorem

- Fundamental fib's  $cod(\mathbb{D}) : \mathbb{D} \downarrow \mathbb{D} \to \mathbb{D}$  of f.l. cat's are extensive
- Extensive fib's are stable under pullback along f.l.p. functors  $\Delta:\mathbb{C}\to\mathbb{D}$
- Thus, gluing fibrations  $gl_{\Delta}(\mathbb{D}) : Gl_{\Delta}(\mathbb{D}) \to \mathbb{C}$  are extensive

*Theorem (Moens' theorem)* 

The assignment  $\Delta \mapsto gl_{\Delta}(\mathbb{D}) = \Delta^* cod(\mathbb{D})$  gives rise to a biequivalence

 $\text{ExtFib}(\mathbb{C})\simeq \mathbb{C}/\!\!/ \text{Lex}$ 

between the 2-category  $\mathsf{ExtFib}(\mathbb{C})$  of extensive fibrations on  $\mathbb{C}$  and the pseudo-co-slice 2-category  $\mathbb{C}/\!\!/\mathsf{Lex}$  of f.l. categories under  $\mathbb{C}$ .

### $\text{ExtFib}(\mathbb{C}) \to \mathbb{C}/\!\!/ \text{Lex}$

The functor corresponding to a fibration  $P : \mathbb{X} \to \mathbb{C}$  is given by

 $\Delta: \mathbb{C} \to \mathbb{X}(1) \qquad 1 \xrightarrow{1 \leftrightarrow \cdots \rightarrow C} 1$   $C \mapsto \sum_{c} 1 \qquad C \xrightarrow{} 1$ 

Gluing fibrations for Grothendieck toposes and realizability toposes

• For Grothendieck toposes  $\mathcal{E}$  with geometric morphism  $\Delta \dashv \Gamma : \mathcal{E} \to$ **Set**, we have

 $\operatorname{gl}_{\Delta}(\mathcal{E}) \simeq \operatorname{fam}(\mathcal{E})$ 

- Thus, when studying Grothendieck toposes  $\Delta \dashv \Gamma : \mathcal{E} \rightarrow Set$  relative to a base topos  $\mathcal{S}$ , the fibration  $gl_{\Delta}(\mathcal{E})$  is an adequate substitute for the family fibration
- For realizability toposes with c.o.f.  $\Delta$  : Set  $\rightarrow$  RT(A), the fibrations  $gl_{\Delta}(RT(A))$  and fam(RT(A)) are different
- · We will see just how different

## Gluing and local smallness

#### Theorem

If  $\Delta : S \to \mathcal{E}$  is a f.l.p. functor between toposes, then  $gl_{\Delta}(\mathcal{E})$  is a locally small fibration iff  $\Delta$  has a right adjoint

 Thus, gluing fibrations gl<sub>△</sub>(RT(A)) of realizability toposes are not locally small

We have two ways of looking at realizability toposes

- From the point of view of ordinary CT, toposes **RT**(*A*) are locally small, but not cocomplete
- Viewed as gluing fibrations, they have small sums, but are not locally small

Characterizing Realizability Toposes

### Motivation

- Peter Johnstone pointed out the lack of a 'Giraud style' theorem for realizability toposes
- It seemed easier to characterize the gluing fibrations gl<sub>Δ</sub>(RT(A)) (or equivalently the functors Δ : Set → RT(A)) than the 'bare' toposes
- Fibrationally realizability toposes resemble presheaf toposes

# Moens' theorem for fibered pretoposes

- A pre-stack is a fibration P : X → R on a regular category R where the reindexing functors e<sup>\*</sup> : P(I) → P(J) are full and faithful for all regular epis e : J → I
- All fibrations on Set are pre-stacks with AC, and without still most
- A fibered pretopos is an extensive pre-stack  $P : \mathbb{X} \to \mathbb{R}$  with exact fibers
- $fam(\mathcal{E})$  is a fibered pretopos iff  $\mathcal{E}$  is an  $\infty$ -pretopos

Theorem (Moens' theorem for fibered pretoposes)

The assignment  $\Delta \mapsto gl(\Delta)$  gives rise to a biequivalence

 $\textbf{PretopFib}(\mathbb{R}) \simeq \mathbb{R}/\!\!/ \textbf{Ex}$ 

between the 2-category  $\operatorname{PretopFib}(\mathbb{R})$  of fibered pretoposes on  $\mathbb{R}$  and the pseudo-co-slice 2-category  $\mathbb{R}/\!\!/\operatorname{Ex}$  of exact categories under  $\mathbb{C}$ .

# Fibered presheaf construction

Theorem Let  $\mathbb{R}$  be a regular category The forgetful functor  $\operatorname{PretopFib}(\mathbb{R}) \to \operatorname{Lex}(\mathbb{R}),$ where  $\operatorname{Lex}(\mathbb{R})$  is the category of finite-limit pre-stacks on  $\mathbb{R}$ , has a left biadjoint  $\mathscr{C} \mapsto \widehat{\mathscr{C}}$ , called fibered presheaf construction.

- If  $\mathbb{C}$  is a small category with finite limits, then  $\widehat{fam}(\mathbb{C}) = fam(\mathbf{Set}^{\mathbb{C}^{op}})$
- For any PCA  $\mathcal{A}$  we have  $\widehat{fam(\mathcal{A})} = gl_{\Delta}(\mathbf{RT}(\mathcal{A}))$

# Characterization of fibrations of presheaves

Which fibered pretoposes  $P : \mathbb{X} \to \mathbb{R}$  are of the form  $\mathscr{X} \simeq \widehat{\mathscr{C}}$ ?

#### Theorem (Bunge 77)

A locally small  $\infty$ -pretopos  $\mathcal{E}$  is a presheaf topos iff it has a separating family of **indecomposable projective** objects.

In a similar way, we can show:

#### Theorem

A fibered pretopos  $\mathscr{X} : |\mathscr{X}| \to \mathbb{R}$  is a fibration of presheaves iff

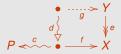
- the subfibration of  ${\mathscr X}$  on indecomposable projectives is closed under finite limits, and
- Every X ∈ |𝔅| can be covered by an internal sum of indecomposable projectives.

 $\ldots$  where indecomposable projectives in fibrations are defined on the next slide

Indecomposables and projectives Let  $\mathscr{X} : |\mathscr{X}| \to \mathbb{R}$  be a fibered pretopos.

Definition

• Call  $P \in |\mathscr{X}|$  projective, if given c, e, f as in the diagram



where *c* is cartesian and *e* is vertical and a regular epimorphism in its fiber, we can fill in *d*, *g* with *d* epicartesian such that the square commutes.

• Call  $X \in |\mathscr{X}|$  indecomposable, if for every diagram



in  $|\mathscr{X}|$  where *c* is cartesian and *d* is cocartesian, there exists a *unique* mediating arrow *m*.

# Characterizing fibered realizability toposes

With a bit of work one can prove the following

Theorem

Gluing fibrations  $gl_{\Delta}(\mathbf{RT}(\mathcal{A}))$  of realizability toposes can be characterized as fibered pretoposes  $P : \mathbb{X} \to \mathbf{Set}$  such that

- P is a fibered cocompletion (previous theorem)
- the fibers of P are lccc
- The subfibration Q ⊆ P on indecomposable projectives is posetal, has a discrete generic predicate, and Q(1) ≃ 1

[discrete means right orthogonal to cartesian maps over surjective functions]

## Characterizing fibered realizability toposes

In realizability toposes, we have  $(\mathbf{RT}(\mathcal{A})(1, -) : \mathbf{RT}(\mathcal{A}) \to \mathbf{Set}) \dashv \Delta$ , thus the global sections functor is uniquely determined and does not contain additional information. Thus, our analysis yields a characterization of 'bare' toposes after all:

#### Theorem

A locally small category  $\mathcal{E}$  is equivalent to a realizability topos  $\mathsf{RT}(\mathcal{A})$  over a PCA  $\mathcal{A}$ , if and only if

- O  $\emph{E}$  is exact and locally cartesian closed,
- 2 E has enough projectives, and the subcategory Proj(E) of projectives is closed under finite limits,
- ④ the global sections functor Γ : E → Set has a right adjoint Δ factoring through Proj(E), and
- ④ there exists a separated and discrete projective  $D \in \mathcal{E}$  such that for all projectives  $P \in \mathcal{E}$  there exists a closed  $u : P \rightarrow D$ .