Bifibrational Parametricity

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Parametricity

Polymorphic functions are functions depending on type variables:

$$f : \forall X . T(X).$$

They can be classified in two classes.

- *Ad hoc polymorphisms.*
  They define different functions for different types. For example $\phi\colon \forall X.X \to X \to X$ can define the sum for natural numbers, concatenetion for lists, . . .

- *Parametric polymorphisms.*
  They define the same function for any type. For example $\mathrm{rev}\colon \forall X.\mathit{list}(X) \to \mathit{list}(X)$ defines the same function independently by the type of the list.

Theorems for Free!

- From parametric polymorphisms is possible to extract properties.
  For example any parametric function

  $$h \colon \forall X. list(X) \to list(X)$$

  satisfies

  $$h(\mathrm{map}\, f(xs)) = \mathrm{map}\, f(h(xs)).$$

- This is thanks to Reynolds' relational interpretation.
- Idea: rev(Nat) and rev(Char) come from the same parametric polymorphism, then they are related.
- Equality relation performs a central role: closed terms are constant polymorphic functions and related with themselves.
- We generalize Reynolds' model using bifibrations.

Notation

- Fibration + Opfibration = Bifibration
- Bifibration $U\colon \mathcal{E} \to \mathcal{B}$.
- The fiber over X is denoted $\mathcal{E}_X$.
- Let $f\colon X \to X'$ be a morphism in $\mathcal{B}$
  - reindexing functor: $f^*\colon \mathcal{E}_{X'} \to \mathcal{E}_X$
  - opreindexing functor: $\Sigma_f\colon \mathcal{E}_X \to \mathcal{E}_{X'}$
- Let $f\colon X \to U(B)$ be a morphism in $\mathcal{B}$, the cartesian morphism is denoted $f^{\S}\colon f^*(B) \to B$.
- Let $f\colon U(A) \to X'$ be a morphism in $\mathcal{B}$, the opcartesian morphism is denoted $f_{\S}\colon A \to \Sigma_f(A)$.

Setting

Definition

The category $\mathtt{Rel}$ *is given by*

- *objects are triples* (A, B, R) *with* A, B *sets and* $R \subseteq A \times B$
- *morphisms* $(f, g)\colon (A, B, R) \to (A', B', R')$ *with* $f\colon A \to A'$, $g\colon B \to B'$ *and they induce* $(f \times g)|_R\colon R \to R'$.

Definition

We define the functor $Rel(U)\colon \mathtt{Rel} \to \mathtt{Set} \times \mathtt{Set}$

- *objects* $Rel(U)(A, B, R) = (A, B)$
- *morphisms* $Rel(U)(f, g) = (f, g)$

The functor $Rel(U)\colon \mathtt{Rel} \to \mathtt{Set} \times \mathtt{Set}$ is a *bifibration*.

Relations Bifibrations

We can generalize the previous setting to

Definition (Relations Bifibration)

Consider a bifibration $U : \mathcal{E} \to \mathcal{B}$ where $\mathcal{B}$ has products. The relations bifibration of $\mathcal{E}$ over $\mathcal{B}$ is $Rel(U)$ arising via change of base

$$\begin{tikzcd} Rel(\mathcal{E}) \arrow[r, "q"] \arrow[d, "Rel(U)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "U"] \\ \mathcal{B} \times \mathcal{B} \arrow[r, "\_\times\_"'] & \mathcal{B} \end{tikzcd}$$

The category $\mathrm{Rel}(\mathcal{E})$ has

- objects: triple (A, B, X) such that $U(X) = A \times B$
- morphisms: triple (f, g, α) such that $U(\alpha) = f \times g$

We can think X as a relation over A and B.

`Rel` is the relations fibration of Sub(`Set`) over `Set`.

Equality Functor

Definition (Truth Functor)

If any fiber $\mathcal{E}_X$ of the fibration $U\colon \mathcal{E} \to \mathcal{B}$ has terminal object $K(X)$, it induces the functor $K\colon \mathcal{B} \to \mathcal{E}$ which is called truth functor.

Definition (Equality Functor)

The *equality functor* $Eq : \mathcal{B} \to \mathtt{Rel}(\mathcal{E})$ for a relations fibration $Rel(U) : Rel(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ is induced by the action on the objects

$$Eq(X) = \Sigma_{\delta_X} KX$$

where $\delta_X : X \to X \times X$ is diagonal morphism.

In $\mathtt{Rel}$ this definition gives the usual notion of equality

$$Eq\ A = \{(a, a) \mid a \in A\}$$

Types Judgements

- A type context Γ is a set of type variables $X_1, \ldots, X_n$.
- A type judgement is of the form $\Gamma \vdash T$ Type
- Type judgements are defined inductively

$$\frac{X_i \in \Gamma}{\Gamma \vdash X_i \text{ Type}}$$

$$\frac{\Gamma \vdash T \text{ Type} \quad \Gamma \vdash U \text{ Type}}{\Gamma \vdash T \to U \text{ Type}} \qquad \frac{\Gamma, X \vdash T \text{ Type}}{\Gamma \vdash \forall X.T \text{ Type}}$$

We use the more concise notation $\Gamma \vdash T$ for type judgements.

Interpretation of Types

Reynolds interprets a type judgement with two functors

$$[\![\Gamma \vdash T]\!]_1 \colon |\mathtt{Rel}|^n \to \mathtt{Rel}$$

$$[\![\Gamma \vdash T]\!]_0 \colon |\mathtt{Set}|^n \to \mathtt{Set},$$

where $|\_|$ for discrete categories and n is the cardinality of Γ.

The fibrational generalization is with fibred functors

$$\begin{tikzcd}
|Rel(\mathcal{E})|^n \arrow[r, "[\![\Gamma\vdash T]\!]_1"] \arrow[d, "|Rel(U)|^n"'] & Rel(\mathcal{E}) \arrow[d, "Rel(U)"] \\
|\mathcal{B}|^n \times |\mathcal{B}|^n \arrow[r, "[\![\Gamma\vdash T]\!]_0 \times [\![\Gamma\vdash T]\!]_0"'] & \mathcal{B} \times \mathcal{B}
\end{tikzcd}$$

Interpretation of Types in Rel

Reynolds' interpretation:

- Type Variables:

$$[\![\Gamma \vdash X_i]\!]_0 \bar{A} \;=\; A_i \quad \text{and} \quad [\![\Gamma \vdash X_i]\!]_1 \bar{R} \;=\; R_i$$

- Arrow Types: we use exponential objects

$$\begin{array}{rcl} [\![\Gamma \vdash T \to U]\!]_0 \bar{A} & = & [\![\Gamma \vdash T]\!]_0 \bar{A} \to [\![\Gamma \vdash U]\!]_0 \bar{A} \\ [\![\Gamma \vdash T \to U]\!]_1 \bar{R} & = & \{(f,g) \mid (a,b) \in [\![\Gamma \vdash T]\!]_1 \bar{R} \Rightarrow \\ & & \qquad (fa, gb) \in [\![\Gamma \vdash U]\!]_1 \bar{R}\} \end{array}$$

Note that they are both equality preserving (IEL later)

$$[\![\Gamma \vdash T]\!]_1(\mathrm{Eq}^n\ \bar{A}) \to [\![\Gamma \vdash U]\!]_1(\mathrm{Eq}^n\ \bar{A}) \cong \mathrm{Eq}([\![\Gamma \vdash T \to U]\!]_0 \bar{A})$$

Forall Types

- `Set`

$$\begin{aligned}\llbracket \Gamma \vdash \forall X.T \rrbracket_0 \bar{A} \quad &= \\ &\{ f : (S : Set) \to \llbracket \Gamma, X \vdash T \rrbracket_0 (\bar{A}, S) \mid \\ &\quad (A_{n+1}, B_{n+1}, R_{n+1}) \in \texttt{Rel} \Rightarrow \\ &\quad\quad (fA_{n+1}, fB_{n+1}) \in \llbracket \Gamma, X \vdash T \rrbracket_1 (Eq^n\ \bar{A}, R_{n+1}) \}\end{aligned}$$

  - ad-hoc functions with an uniformity condition

- `Rel`

$$\begin{aligned}\llbracket \Gamma \vdash \forall X.T \rrbracket_1 \bar{R} \quad &= \\ &\{ (f, g) \mid (A_{n+1}, B_{n+1}, R_{n+1}) \in \texttt{Rel} \Rightarrow \\ &\quad (fA_{n+1}, gB_{n+1}) \in \llbracket \Gamma, X \vdash T \rrbracket_1 (\bar{R}, R_{n+1}) \}\end{aligned}$$

  - maps are related iff they map related inputs to related outputs

Interpretation of Types Fibrationally

Fibrational interpretation:

- Type Variables:

$$[\![\Gamma \vdash X_i]\!]_0 \bar{A} \;=\; A_i \quad \text{and} \quad [\![\Gamma \vdash X_i]\!]_1 \bar{R} \;=\; R_i$$

- Arrow Types: another time exponential objects

$$\begin{aligned} [\![\Gamma \vdash T \to U]\!]_0 \bar{A} &= [\![\Gamma \vdash T]\!]_0 \bar{A} \to [\![\Gamma \vdash U]\!]_0 \bar{A} \\ [\![\Gamma \vdash T \to U]\!]_1 \bar{R} &= [\![\Gamma \vdash T]\!]_1 \bar{R} \to [\![\Gamma \vdash U]\!]_1 \bar{R} \end{aligned}$$

It's important that they are equality preserving (IEL later)

$$[\![\Gamma \vdash T]\!]_1 (\mathrm{Eq}^n\ \bar{A}) \to [\![\Gamma \vdash U]\!]_1 (\mathrm{Eq}^n\ \bar{A}) \cong \mathrm{Eq}([\![\Gamma \vdash T \to U]\!]_0 \bar{A})$$

Forall Types Fibrationally

Usually universal quantifier right adjoint of projection.

It is important the fibred structure: the adjunction is influenced by the mutually dependence between base and total category.

Define

$$|Rel(U)|^n \to_{Eq} Rel(U)$$

to be the category of equality preserving fibred functors and fibred natural transformations between them.

We interpret $\forall$ as the right adjoint of the projection functor

$$\_ \circ \pi : (|Rel(U)|^n \to_{Eq} Rel(U)) \to (|Rel(U)|^{n+1} \to_{Eq} Rel(U))$$

Key Theorems in Reynolds' Model

Reynolds' work is based on two important results:

- **Identity Extension Lemma**, which asserts that equality commutes with the interpretation of types
- **Abstraction Theorem**, which asserts that the interpretation gives maps between relations which preserve relations, i.e. related elements are sent to related elements.

We generalize these results to the fibrational setting and compare the statements in Reynolds' style and in the fibrational one.

Identity Extension Lemma

Lemma (IEL, Reynolds-style)

If $\Gamma \vdash T$, then for every object $\bar{A}$ in $\mathcal{S}et^n$

$$[\![\Gamma \vdash T]\!]_1(Eq^n\,\bar{A}) = Eq\,([\![\Gamma \vdash T]\!]_0\bar{A})$$

The formulation of the IEL fibrational is:

Lemma (IEL, Fibrationally)

If $\Gamma \vdash T$ then $[\![T]\!]$ is equality preserving, i.e., the following diagram commutes:

$$\begin{tikzcd} |Rel(\mathcal{E})|^n \arrow[r, "{[\![\Gamma\vdash T]\!]_1}"] & Rel(\mathcal{E}) \\ |\mathcal{B}|^n \arrow[u, "|Eq|^n"] \arrow[r, "{[\![\Gamma\vdash T]\!]_0}"'] & \mathcal{B} \arrow[u, "Eq"'] \end{tikzcd}$$

Term Context

A *term context* is of the form $\Gamma \vdash \Delta$, where

- Γ is a type context
- Δ is of the form $x_1 : T_1, ..., x_m : T_m$
- $\Gamma \vdash T_i$ Type for any $i \in \{1, \dots, m\}$

Reynolds interprets term context as pointwise product of types

$$[\![\Gamma \vdash \Delta]\!]_0 = [\![\Gamma \vdash T_1]\!]_0 \times \cdots \times [\![\Gamma \vdash T_n]\!]_0$$

and

$$[\![\Gamma \vdash \Delta]\!]_1 = [\![\Gamma \vdash T_1]\!]_1 \times \cdots \times [\![\Gamma \vdash T_n]\!]_1.$$

Given a relations fibration $Rel(U)$ in which both the categories have products, it is easy to generalize to the fibred functor $[\![\Gamma \vdash \Delta]\!]$.

Reynolds' Interpretation of Terms

Reynolds' set-valued semantics of terms uses induction on the structure of term judgements to give, for each judgement $\Gamma, \Delta \vdash t : T$, a function

$$[\![\Gamma, \Delta \vdash t : T]\!]_0 : (\bar{A} : |\mathtt{Set}|^n) \to [\![\Gamma \vdash \Delta]\!]_0 \bar{A} \to [\![\Gamma \vdash T]\!]_0 \bar{A}.$$

It is like we have a map from $[\![\Gamma \vdash \Delta]\!]_0$ to $[\![\Gamma \vdash T]\!]_0$ in the environment $\bar{A}$.

They can be defined by induction on terms in the expected way.

Terms are Natural Transformations

For every object $\bar{A}$ of $\mathtt{Set}^n$, in $\mathtt{Set}$ we have a morphism

$$[\![\Gamma \vdash \Delta]\!]_0 \bar{A} \to [\![\Gamma \vdash T]\!]_0 \bar{A}$$

Both $[\![\Gamma \vdash \Delta]\!]_0$ and $[\![\Gamma \vdash T]\!]_0$ define functors $|\mathtt{Set}|^n \to \mathtt{Set}$.
We have natural transformation

$$[\![\Gamma, \Delta \vdash t : T]\!]_0 : [\![\Gamma \vdash \Delta]\!]_0 \to [\![\Gamma \vdash T]\!]_0.$$

We write $[\![\Delta]\!]_0$ for $[\![\Gamma \vdash \Delta]\!]_0$, and $[\![t]\!]_0$ for $[\![\Gamma, \Delta \vdash t : T]\!]_0$ and we can generalize to the fibrational setting as the natural transformation

$$\begin{tikzcd} |\mathcal{B}|^n \arrow[r, bend left, "{[\![\Delta]\!]_0}"] \arrow[r, bend right, "{[\![T]\!]_0}"'] \arrow[r, phantom, "\Downarrow {[\![t]\!]_0}"] & \mathcal{B} \end{tikzcd}$$

Abstraction Theorem

Theorem (Abstraction Theorem, Reynolds-Style)

Let $\bar{A}, \bar{B} \in \mathtt{Set}^n$, $\bar{R} \in \mathtt{Rel}^n(\bar{A}, \bar{B})$, $(a, b) \in [\![\Gamma \vdash \Delta]\!]_1 \bar{R}$ *then*

$$([\![t]\!]_0\ \bar{A}\ a, [\![t]\!]_0\ \bar{B}\ b) \ \in\ [\![\Gamma \vdash T]\!]_1 \bar{R}$$

Theorem (Abstraction Theorem, Fibrationally)

Every judgement $\Gamma, \Delta \vdash t : T$ *defines a fibred natural transformation* $[\![t]\!] : [\![\Gamma \vdash \Delta]\!] \to [\![\Gamma \vdash T]\!]$

$$\begin{tikzcd}
{|Rel(\mathcal{E})|^n} \arrow[rr, bend left, "{[\![\Delta]\!]_1}"] \arrow[rr, bend right, "{[\![T]\!]_1}"'] \arrow[rr, phantom, "\Downarrow [\![t]\!]_1"] \arrow[d, "U^n"'] & & Rel(\mathcal{E}) \arrow[d, "U"] \\
{|\mathcal{B}|^n \times |\mathcal{B}|^n} \arrow[rr, bend left, "{[\![\Delta]\!]_0 \times [\![\Delta]\!]_0}"] \arrow[rr, bend right, "{[\![T]\!]_0 \times [\![T]\!]_0}"'] \arrow[rr, phantom, "\Downarrow [\![t]\!]_0 \times [\![t]\!]_0"] & & \mathcal{B} \times \mathcal{B}.
\end{tikzcd}$$

And the Morphisms?

- Some types define mixvariant functor, for this reason we use discrete domains
- By using discrete categories we forget about morphisms
- For any morphism $f : X \to Y$ between sets there is the graph relation $\langle f \rangle = \{(x, y) \mid fx = y\} \subseteq X \times Y$
- We can generalize this definition of graph relation
- In this way we replace the action of $[\![\Gamma \vdash T]\!]_0$ on morphisms with the action of $[\![\Gamma \vdash T]\!]_1$ on graph relations

Graph Functor: Fibrational or Opfibrational?

Let $f\colon X \to Y$ be a morphism in $\mathcal{B}$.

We have two candidate functors $\langle \_ \rangle, \langle \_ \rangle' \colon \mathcal{B}^{\to} \to Rel(\mathcal{E})$

- objects:
  - $\langle f \rangle = \Sigma_{(id_X, f)}(Eq\, X)$
  - $\langle f \rangle' = (f, id_Y)^*(Eq\, Y)$
- morphisms:

$$
\begin{tikzcd}
 & & X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] & & \\
 & & X' \arrow[r, "f'"'] \arrow[dl, "\langle \_ \rangle"'] & Y' \arrow[dr, "\langle \_ \rangle'"] & & \\
Eq(X) \arrow[r, "(id_X, f)_{\S}"] \arrow[d, "Eq(\alpha)"'] & \langle f \rangle \arrow[d, dashed, "\langle (\alpha, \beta) \rangle"] & & & \langle f \rangle \arrow[r, "(f, id_Y)^{\S}"] \arrow[d, dashed, "\langle (\alpha, \beta) \rangle'"'] & Eq(Y) \arrow[d, "Eq(\alpha)"] \\
Eq(X') \arrow[r, "(id_{X'}, f')_{\S}"] & \langle f' \rangle & & & \langle f' \rangle \arrow[r, "(f', id_{Y'})^{\S}"] & Eq(Y').
\end{tikzcd}
$$

Graph Functor: Some Properties

Lemma

In our model the two definitions are equivalent.

The proof is based on Beck–Chevalley condition.
We can use two universal properties: cartesian and opcartesian.

Note that equality is a particular case of graph relation for identity morphism, in fact we have

$$\mathrm{Eq}(X) \cong \langle id_X \rangle.$$

Lemma

If Eq is full and faithful then the graph functor is full and faithful.

A Taste of Bifibrationality: Graph Lemma

Lemma (Graph Lemma, Reynolds-Style)

For a syntactic functor F, $F_1\langle h\rangle = \langle F_0(h)\rangle$

Lemma

For a fibred functor F there are two vertical morphisms

$$\psi\colon \langle F_0 h\rangle \to F_1(\langle h\rangle) \qquad\qquad \phi\colon F_1(\langle h\rangle') \to \langle F_0 h\rangle'$$

such that $\phi \circ \psi = id$.

$$\begin{tikzcd}
F_1(Eq(X)) \arrow[r, "F_1((id,h)_\S)"] \arrow[d, "\cong"'] & F_1(\langle h\rangle) \arrow[r, "\cong"] & F_1(\langle h\rangle') \arrow[r, "F_1((h,id)^*)"] \arrow[d, dashed, "\phi"] & F_1(Eq(Y)) \arrow[d, "\cong"] \\
Eq(F_0X) \arrow[r, "(id,F_0h)_\S"'] & \langle F_0h\rangle \arrow[r, "\cong"'] \arrow[u, dashed, "\psi"] & \langle F_0h\rangle' \arrow[r, "(F_0h,id)^*"'] & Eq(F_0Y)
\end{tikzcd}$$

The results shown as far about the graph functor are used to prove that $[\![\forall X.(TX \to X) \to X]\!]$ is the carrier of the initial $[\![T]\!]_0$-algebra.

Related Work

- Dunphy and Reddy *"Parametric limits"*
- Hermida, Reddy and Robinson *"Logical Relations and Parametricity - A Reynolds Programme for Category Theory and Programming Languages"*
- Birkedal, Møgelberg *"Categorical Models of Parametric Polymorphism"*
- Some unpublished works of Hermida

Future Work

- Extension to intensional Martin-Löf's Type Theory
- Higher Order Parametricity

$$
\begin{tikzcd}
\mathsf{Rel}(\mathcal{C}) \arrow[r] \arrow[d, "\mathsf{Rel}(P)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "P"] & \\
\mathsf{Rel}(\mathcal{E}) \times \mathsf{Rel}(\mathcal{E}) \arrow[r, "\_\times\_"'] \arrow[d] & \mathsf{Rel}(\mathcal{E}) \arrow[r] \arrow[d, "\mathsf{Rel}(U)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \mathcal{B} \arrow[r, "(\_\times\_{,}\_\times\_)"'] & \mathcal{B} \times \mathcal{B} \arrow[r, "\_\times\_"'] & \mathcal{B}
\end{tikzcd}
$$

$\mathcal{B} = \mathtt{Set}$ $\quad$ $\mathcal{E} = \mathsf{Fam}(\mathtt{Set})$ $\quad$ $\mathcal{C} = \mathsf{Sub}(\mathsf{Rel}(\mathsf{Fam}(\mathtt{Set})))$