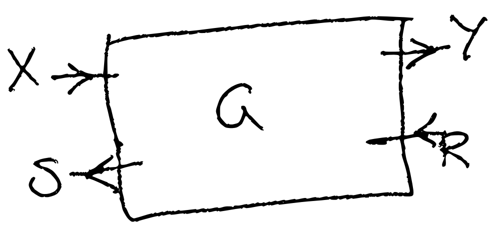
Recall

An open game

$G : (X, S) \longrightarrow (Y, R)$



is (1) sets X, S, Y, R of state, coutility, moves & utility

(2) A set Σ of strategies

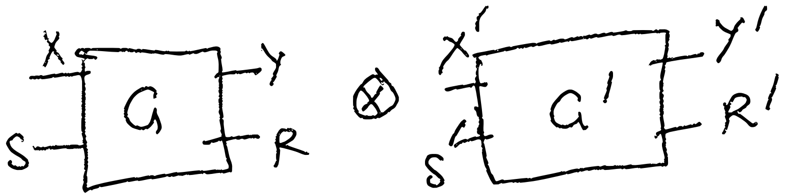
(3) $P : \Sigma \times X \to Y$

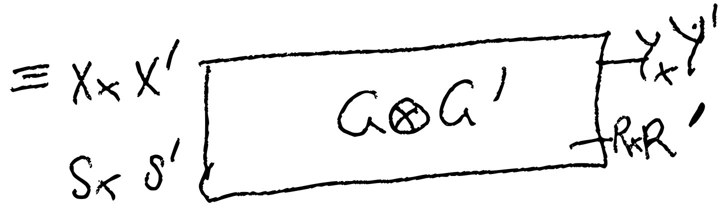
(4) $C : \Sigma \times X \times R \longrightarrow S$

(5) $E : X \times (Y \to R) \longrightarrow \mathbb{P}\Sigma$

Open Games

- $\parallel$ composition





with strategies $\Sigma \times \Sigma'$

- $P_{G \otimes G'} : (\Sigma \times \Sigma') \times (X \times X') \to (Y \times Y')$

  $P_{G \otimes G'}(\sigma, \sigma')\,(x, x') = (P_G\, \sigma\, x,\ P_{G'}\, \sigma'\, x')$

- $C_{G \otimes G'} : (\Sigma \times \Sigma') \times (X \times X') \times (R \times R') \to S \times S'$

  $C_{G \otimes G'}\,(\sigma, \sigma')\,(x, x')\,(r, r') = (C_G\, \sigma\, x\, r,\ C_{G'}\, \sigma'\, x'\, r')$

- $E_{G \otimes G'} : (X \times X') \times (Y \times Y' \to R \times R')$
  $\to P(\Sigma \times \Sigma')$

$(\sigma, \sigma') \in E_{G \otimes G'}\ (x, x')\ k$ iff

① $\sigma \in E_G\ x\ (\underbrace{\lambda y \to \pi_0\, k(y, P_{G'}\, x' \sigma')}_{: Y \to R})$

&

$\sigma' \in E_{G'}\ x'\ (\underbrace{\lambda y' \to \pi_1\, k(P_G\, x \sigma, y')}_{: Y' \to R'})$

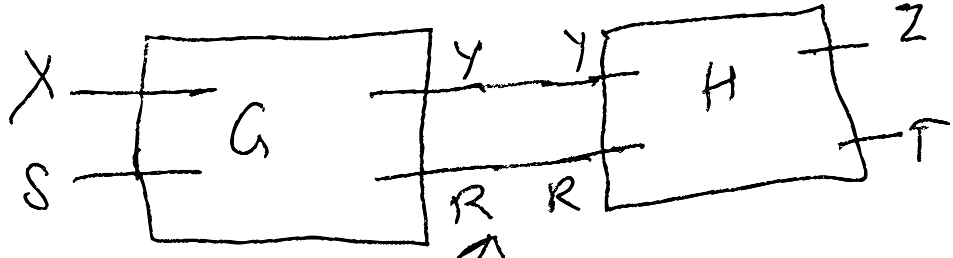
* Compositional definition
* Like for simple games
* Nash is a special case

Sequential Composition

- This motivated introduction of coutility. Remember bookkeeper & better example

<u>Types first</u>

$$\frac{\begin{array}{l} G : (X,S) \to (Y,R) \\ H : (Y,R) \to (Z,T) \end{array}}{H \circ G : (X,S) \longrightarrow (Z,T)}$$



the utility for G supplied via H coutility

Formally

Strategies of $H.G = \Sigma_H \times \Sigma_G'$

again product.

$$P_{H.G} : (\Sigma_H \times \Sigma_G) \times X \longrightarrow Z$$

$$P_{H.G}\,(\sigma';\sigma)\;x = P_H\,\sigma'\,(P_G\,\sigma\,x)$$

$$C_{H.G} : (\Sigma_H \times \Sigma_G) \times X \times T \longrightarrow S$$

$$C_{H.G}\,(\sigma';\sigma)\;x\;t =$$
$$C_G\,\sigma\,x\,(C_H\,\sigma'\,(P_G\,x\,\sigma)\,t)$$

Finally Core part.

$$E_{H.G} : X \times (Z \to T) \to \mathbb{P}(\Sigma_H \times \Sigma_G)$$

$$(\sigma', \sigma) \in E_{H.G} \; x \; (k : Z \to T)$$

iff

$$\sigma \in E_G \; x \; (\lambda y \to C_H \, \sigma' \, y \, (k(P_H \sigma' y)))$$

$$\sigma' \in E_H \; (P_G \, \sigma \, x) \qquad k$$

uses C_H to $: Y \to R$
return H-coutility
to G as utility & hence
create G's utility function.

Great!!!

Model of compositional game theory with 2 operators for building complex equilibria from simpler.

What else

1. More operators
   - choice
   - iteration
   - subgame perfection

2. Probability & other effects

3. Software

4. Machine Learning

5. You tell me

Category Theory

- Problem: Def of H-G & $G \otimes G'$ is complex as structures are 8-tuples
    - simple proofs ought to be easy but are not, eg assoc of $\otimes$ & $\circ$
    - complex proofs will become untractable

Ans Need some abstraction

Category theory

Key idea Often we have arrows with source & target $A \xrightarrow{f} B$. There is an assoc operator for composing arrows with a unit.

$\Rightarrow$ This is all a category is

* A, B etc are called objects

$A \xrightarrow{f} B$ could be

<u>logic</u> : f is a proof of B assuming A

<u>programming</u> : f is a program producing data of type B using data of type A

<u>algebra</u> : f is a homomorphism from some algebraic structure A to an algebraic structure B (eg group homom.)

<u>maths</u> : f is some function between sets/preorders/spaces A & B

Defn

A category $\mathbb{C}$ is

(i) a set of objects $|\mathbb{C}|$

(ii) for each pair of objects $A, B \in |\mathbb{C}|$, a set of arrows

$$\mathbb{C}(A,B)$$

(iii) for each triple $A, B, C \in |\mathbb{C}|$, a composition

$$\_\circ\_ : \mathbb{C}(B,C) \times \mathbb{C}(A,B) \longrightarrow \mathbb{C}(A,C)$$

(iv) for each object $A \in |\mathbb{C}|$, an identity $1_A \in \mathbb{C}(A,A)$

st comp is assoc & 1 is the unit

Here is a category

(1) The category of Sets

$A \xrightarrow{f} B$ is a function

(2) Lenses: objects are pairs of sets

arrows

$(X,S) \longrightarrow (Y,R)$ are defined to be

a function $X \longrightarrow Y$

& a function $X \times R \longrightarrow S$

... we are using lenses all the time. There is a forward & backward feel

eg continuity, back propagation

Thm Lenses form a category

Pf $(X,S) \xrightarrow{1} (X,S)$

(i) $\triangleq$ $X \longrightarrow X$ identity

$X \times S \longrightarrow S$ second projection

(ii) $(X,S) \longrightarrow (Y,R)$ $\quad$ $(Y,R) \longrightarrow (Z,T)$

$f : X \longrightarrow Y$ $\quad$ $g : Y \longrightarrow Z$

$f' : X \times R \longrightarrow S$ $\quad$ $g' : Y \times T \longrightarrow R$

composite $(X,S) \longrightarrow (Z,T)$

needs $\begin{cases} \text{first function} & X \xrightarrow{f} Y \xrightarrow{g} Z \\ \text{second function} & X \times T \longrightarrow S \end{cases}$

$$\begin{tikzcd}
X \times T \arrow[rrr] \arrow[d, "\Delta \times 1"] & & & S \\
X \times X \times T \arrow[r, "1 \times f \times 1"'] & X \times Y \times T \arrow[rr, "1 \times g'"'] & & X \times R \arrow[u, "f'"']
\end{tikzcd}$$

or $\lambda x t \longrightarrow f'(x, g'(fx, t))$