PROOF-THEORETIC SEMANTICS FOR DYNAMIC LOGICS

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Proof-Theoretic Semantics

Theories of meaning	
Denotational (model-theoretic)	Inferential (proof-theoretic)
Tarski: Meaning is out there	Gentzen: Meaning is in Rules

- Wittgenstein: meaning is use (very influential in philosophy of language)
- Wansing: meaning is **correct** use!
- not all proof systems are good environments for an inferential theory of meaning.

Good Proof Systems for DLs: Desiderata

- An **independent** account of dynamic logics:
  - Proof-theoretic semantic approach;
- Intuitive, **user-friendly** rules;
- **Good performances**:
  - soundness & completeness,
  - cut-elimination & sub-formula property,
  - decidability.
- A **modular** account of dynamic logics:
  - charting the space of DLs by adding/subtracting rules,
  - transfer of results with minimal changes.

Problems: the case study of DEL

$$\langle\alpha\rangle p \leftrightarrow Pre(\alpha) \wedge p$$
$$\langle\alpha\rangle(A \vee B) \leftrightarrow \langle\alpha\rangle A \vee \langle\alpha\rangle B$$
$$\langle\alpha\rangle\neg A \leftrightarrow Pre(\alpha) \wedge \neg\langle\alpha\rangle A$$
$$\langle\alpha\rangle\langle \mathsf{a}\rangle A \leftrightarrow Pre(\alpha) \wedge \bigvee\{\langle \mathsf{a}\rangle\langle\beta\rangle A \mid \alpha \mathsf{a} \beta\}$$

1. not closed under **uniform substitution**;
2. use of **meta-linguistic abbreviation** $Pre(\alpha)$;
3. use of **labels** $\alpha \mathsf{a} \beta$.

The case study of PDL

$$[\alpha]\,(A \to B) \to ([\alpha]\,A \to [\alpha]\,B)$$

$$[\alpha \cup \beta]\,A \leftrightarrow [\alpha]\,A \wedge [\beta]\,A$$

$$[\alpha\,;\beta]\,A \leftrightarrow [\alpha][\beta]\,A$$

$$[?A]\,B \leftrightarrow (A \to B)$$

$$[\alpha]\,(A \wedge B) \leftrightarrow [\alpha]\,A \wedge [\alpha]\,B$$

$$[\alpha^*]\,A \leftrightarrow A \wedge [\alpha]\,[\alpha^*]\,A$$

$$A \wedge [\alpha^*]\,(A \to [\alpha]\,A) \to [\alpha^*]\,A$$

Display Calculi

- Natural generalization of sequent calculi;
- sequents $X \vdash Y$, where X, Y structures:
  $\phi \quad , \quad \phi ; \psi \qquad \ldots, X > Y, \ldots$
- Display property:

$$\cfrac{\cfrac{\cfrac{Y \vdash X > Z}{X ; Y \vdash Z}}{Y ; X \vdash Z}}{X \vdash Y > Z}$$

- display property: **adjunction** at the structural level.
- Canonical proof of cut elimination

More on structural connectives

- One for two:

$>$		$;$		I		$\{\mathtt{a}\}$		$\underbrace{\overbrace{\mathtt{a}}}$		$\{\alpha\}$		$\underbrace{\overbrace{\alpha}}$	
$\succ$	$\rightarrow$	$\wedge$	$\vee$	$\top$	$\bot$	$\langle\mathtt{a}\rangle$	$[\mathtt{a}]$	$\widecheck{\widehat{\mathtt{a}}}$	$\underline{\overline{\mathtt{a}}}$	$\langle\alpha\rangle$	$[\alpha]$	$\widecheck{\widehat{\alpha}}$	$\underline{\overline{\alpha}}$

- Again, dynamic adjoints needed for display rules:

$$\frac{X \vdash \{\mathtt{a}\} Y}{\underbrace{\overbrace{\mathtt{a}}} X \vdash Y} \qquad \frac{\{\mathtt{a}\} X \vdash Y}{X \vdash \underbrace{\overbrace{\mathtt{a}}} Y}$$

$$\frac{X \vdash \{\alpha\} Y}{\underbrace{\overbrace{\alpha}} X \vdash Y} \qquad \frac{\{\alpha\} X \vdash Y}{X \vdash \underbrace{\overbrace{\alpha}} Y}$$

The multi-type approach

- Ag Act Fnc Fm;
  - no ancillary symbols; all types are **first-class citizens**;
- Additional expressivity:
  - operational connectives **merging different types**:

$$
\begin{array}{lcll}
\vartriangle_1, \blacktriangle_1 & : & \mathsf{Act} \times \mathsf{Fm} \to \mathsf{Fm} & \langle\alpha\rangle A \rightsquigarrow \alpha \vartriangle_1 A \\
\vartriangle_2, \blacktriangle_2 & : & \mathsf{Ag} \times \mathsf{Fm} \to \mathsf{Fm} & \langle \mathsf{a} \rangle A \rightsquigarrow \mathsf{a} \vartriangle_2 A \\
\vartriangle_3, \blacktriangle_3 & : & \mathsf{Ag} \times \mathsf{Fnc} \to \mathsf{Act} &
\end{array}
$$

- Modularity: by adding or subtracting types (Games, strategies, coalitions) one can chart the whole space of dynamic logics.

for $1 \leq i \leq 3$,

	$\vartriangle_i$		$\blacktriangle_i$		$-\!\!\vartriangleright_i$		$-\!\!\blacktriangleright_i$	
	$\vartriangle_i$		$\blacktriangle_i$			$-\!\!\vartriangleright_i$		$-\!\!\blacktriangleright_i$

A glimpse at rules for DEL

Single-type, first version: formulas as side conditions (and rules with labels);

$$\textit{swap-in}_L \; \frac{Pre(\alpha)\,;\{\alpha\}\{\mathtt{a}\}X \vdash Y}{Pre(\alpha)\,;\{\mathtt{a}\}\{\beta\}_{\alpha\mathtt{a}\beta}X \vdash Y}$$

Single-type, emended: purely structural (but labels still there);

$$\textit{swap-in}'_L \; \frac{\{\alpha\}\{\mathtt{a}\}X \vdash Y}{\Phi_\alpha;\{\mathtt{a}\}\{\beta\}_{\alpha\mathtt{a}\beta}X \vdash Y}$$

Multi-type: no side conditions and no labels.

$$\text{swap-in}_L \; \frac{\mathtt{a}\blacktriangle_2(\alpha\blacktriangle_1 X) \vdash Y}{(\mathtt{a}\blacktriangle_3\alpha)\blacktriangle_1(\mathtt{a}\blacktriangle_2 X) \vdash Y}$$

A glimpse at rules for PDL

$$\oplus\ominus\ \frac{\Pi^{\oplus} \vdash \Delta}{\Pi \vdash \Delta^{\ominus}}$$

$$\omega\ \triangle\ \frac{\left(\ \Pi^{(n)} \triangle_1 X \vdash Y\ \middle|\ n \geq 1\ \right)}{\Pi^{\oplus} \triangle_0 X \vdash Y}$$

Canonical cut elimination, 1/3

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined congruence:
   - same shape, same position, same type, non-proliferation;
3. **principal** = **displayed** (Exception: principal fma's in axioms)
   - Generaliz.: axioms are **closed** under display rules (when applicable);
4. rules are closed under **uniform substitution** of congruent parameters within each type;
5. **reduction strategy** exists when cut formulas are both principal.

   Specific to multi-type setting:

6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

Thm: For any (multi-type) calculus satisfying list above, the cut elimination theorem can be proven.

Canonical cut elimination, 2/3

Two main cases + subcases.
(a) Both cut formulas are principal. by 5. (cut is either eliminated or "broken down" into cuts of lower rank).
(b) At least one cut formula is parametric. Subcase (b1): a_u principal in axiom. Then,

$$(x' \vdash y')[a_u^{pre}, a_{suc}]$$

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ x \vdash a\end{array} \qquad \begin{array}{c}\vdots\ \pi_2\\ a \vdash y\end{array}}{x \vdash y} \quad \rightsquigarrow \quad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\ \pi_1\\ x \vdash a\end{array} \qquad a_u \vdash y''[a_{suc}]}{x \vdash y''[a_{suc}]}\\ \vdots\ \pi''\\ (x' \vdash y')[x^{pre}, a_{suc}]\\ \vdots\ \pi_2[x/a_u]\\ x \vdash y\end{array}$$

Canonical cut elimination, 3/3

Subcase (b2): a_u principal in other rule. Then, a_u is in display, and hence:

$$
\dfrac{\begin{array}{c} \vdots\, \pi_1 \\ x \vdash a \end{array} \qquad \begin{array}{c} \vdots\, \pi_2' \\ a_u \vdash y' \\ \vdots\, \pi_2 \\ a \vdash y \end{array}}{x \vdash y}
\quad \rightsquigarrow \quad
\begin{array}{c} \dfrac{\begin{array}{c} \vdots\, \pi_1 \\ x \vdash a \end{array} \qquad \begin{array}{c} \vdots\, \pi_2' \\ a_u \vdash y' \end{array}}{x \vdash y'} \\ \vdots\, \pi_2[x/a] \\ x \vdash y \end{array}
$$

Canonical cut elimination, 3/3

Subcase (b2): a_u principal in other rule. Then, a_u is in display, and hence:

$$\dfrac{\dfrac{\vdots\ \pi_1}{x \vdash a} \qquad \dfrac{\dfrac{\vdots\ \pi_2'}{a_u \vdash y'}}{\dfrac{\vdots\ \pi_2}{a \vdash y}}}{x \vdash y} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots\ \pi_1}{x \vdash a} \qquad \dfrac{\vdots\ \pi_2'}{a_u \vdash y'}}{x \vdash y'}}{\dfrac{\vdots\ \pi_2[x/a]}{x \vdash y}}$$

Subcase (b3): a_u parametric. Then:

$$\dfrac{\dfrac{\vdots\ \pi_1}{x \vdash a} \qquad \dfrac{\dfrac{\vdots\ \pi_2'}{(x' \vdash y')[a_u]^{pre}}}{\dfrac{\vdots\ \pi_2}{a \vdash y}}}{x \vdash y} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdots\ \pi_2'}{(x' \vdash y')[x/a_u^{pre}]}}{\dfrac{\vdots\ \pi_2[x/a_u^{pre}]}{x \vdash y}}$$

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