Chapter I: Measure Theory

Sigma algebra over X: $\quad \Sigma \subseteq 2^X \quad \begin{cases} \cdot\ X, \emptyset \in \Sigma \\ \cdot\ S \in \Sigma \Rightarrow X \setminus S \in \Sigma \\ \cdot\ (S_i)_{i \in \mathbb{N}} \in \Sigma \Rightarrow \bigcup_i S_i \in \Sigma \end{cases}$

Example 1: 2^X — discrete σ-algebra

Example 2: Borel algebra of a topological space (X, τ)

$$\mathrm{Bo}\,X = \sigma(\tau)$$

Measurable map: $\quad (X, \Sigma) \xrightarrow{f} (Y, \Omega)$

$$\forall S \in \Omega, \quad f^{-1}(S) \in \Sigma$$

Meas – measurable spaces and measurable maps

Measures

$$\mu : (X, \Sigma) \longrightarrow \overline{\mathbb{R}}_+ \quad \begin{cases} \bullet\ \mu(\emptyset) = 0 \\ \bullet\ \mu\left(\bigcup_{i\in\mathbb{N}} E_i\right) = \sum_{i\in\mathbb{N}} \mu(E_i) \\ \quad \text{for } E_i \cap E_j = \emptyset \text{ if } i \neq j \end{cases}$$

subprobabilistic measure

$$\mu : (X, \Sigma) \longrightarrow [0, 1]$$

probabilistic measure

$$\mu(X) = 1$$

Example : Dirac distribution on $x \in X$

$S \in \Sigma$,
$$\delta_x(S) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Discrete probability monad on Set

$$\mathrm{Set} \xrightarrow{\Pi} \mathrm{Set}$$

On objects:

$$A \longmapsto \Pi A = \{ p : A \to [0,1] \,/\, p\text{-distrib. with finite support} \}$$

On morphisms:

$$A \xrightarrow{f} B \qquad\qquad \Pi A \xrightarrow{\Pi f} \Pi B$$

for $p \in \Pi A$, $b \in B$

$$\Pi f(p)(b) = \sum_{a \in f^{-1}(b)} p(a) = p\left(f^{-1}(b)\right)$$

$$\mathbf{Set} \xrightarrow{\Pi} \mathbf{Set}$$

<u>unit</u>: $\eta_A : A \longrightarrow \Pi A$

for $a \in A$, $\eta_A(a) = \delta_a$

<u>multiplication</u>: $\mu_A : \Pi\Pi A \longrightarrow \Pi A$

for $\pi \in \Pi\Pi A$
$a \in A$

$$\mu_A(\pi)(a) = \sum_{p \in \mathrm{supp}(\pi)} \pi(p) \cdot p(a)$$

Discrete probability monad:

(Π, η, μ) is a monad on $\mathbf{Set}$

$EM(\Pi)$ form a variety, hence it must exist an equational axiomatization (a Lawvere theory) for it.

Barycentric algebras (Stone 1949)

Signature: $\{ +_\varepsilon : 2 \mid \varepsilon \in [0,1] \}$

Axioms:

(B_1): $\vdash x +_1 y = x$

(B_2): $\vdash x +_\varepsilon x = x$

(SC): $\vdash x +_\varepsilon y = y +_{1-\varepsilon} x$

(SA): $\vdash (x +_\varepsilon y) +_\delta z = x +_{\varepsilon\delta} (y +_{\frac{\delta - \varepsilon\delta}{1-\varepsilon\delta}} z)$

Interpret $\mu +_\varepsilon \nu = \varepsilon\mu + (1-\varepsilon)\nu$ on ΠA

$(\Pi A, +_\varepsilon) \models \mathcal{B}$. Moreover, $(\Pi A, +_\varepsilon) \cong \mathrm{Free}_{\mathcal{B}}(A)$

$EM(\Pi) \cong \mathrm{Models}(\mathcal{B})$

Giry monad on $\mathbb{M}$eas — Lawvere (1964) unpublished

$$\mathbb{M}\text{eas} \xrightarrow{\Delta} \mathbb{M}\text{eas}$$

On objects: $(X, \Sigma) \Rightarrow \overline{X} = \{\mu : \Sigma \to [0,1] \,/\, \mu \text{ measure}\}$

for $r \in [0,1]$, $S \in \Sigma$ $\quad ev_r(S) = \{\mu \in \overline{X} \,/\, \mu(S) \leq r\}$

$$\Delta(X, \Sigma) = \left(\overline{X}, \sigma(\{ev_r(S) \,/\, r \in [0,1], S \in \Sigma\})\right)$$

Giry monad on $\mathbb{M}\mathrm{eas}$ — Lawvere (1964) unpublished

$$\mathbb{M}\mathrm{eas} \xrightarrow{\Delta} \mathbb{M}\mathrm{eas}$$

On objects: $(X, \Sigma) \implies \overline{X} = \{\mu : \Sigma \to [0,1] \,/\, \mu \text{ measure}\}$

for $r \in [0,1]$, $S \in \Sigma$ $\quad ev_r(S) = \{\mu \in \overline{X} \,/\, \mu(S) \leq r\}$

$$\Delta(X, \Sigma) = (\overline{X}, \sigma(\{ev_r(S) \,/\, r \in [0,1], S \in \Sigma\}))$$

On morphisms: $(X, \Sigma) \xrightarrow{f} (Y, \Omega) \implies \Delta(X, \Sigma) \xrightarrow{\Delta f} \Delta(Y, \Omega)$

for $\mu \in \Delta(X, \Sigma)$, $B \in \Omega$

$$\Delta f(\mu)(B) = \mu(f^{-1}(B))$$

$$\mathrm{Meas} \xrightarrow{\Delta} \mathrm{Meas}$$

<u>unit</u>: $\eta_A : A \longrightarrow \Delta A$

for $a \in A$, $\eta_A(a) = \delta_a$

<u>multiplication</u>: $\mu_A : \Delta\Delta A \longrightarrow \Delta A$

for $\pi \in \Delta\Delta A$, $S \in \Sigma$

$$\mu_A(\pi)(S) = \int_{\Delta A} \rho(S) \cdot \pi(d\rho)$$

Giry monad on Meas : (Δ, η, μ)

Chapter II : Metric Spaces

$$d : X \times X \longrightarrow \overline{\mathbb{R}_+}$$

metric: (i) $d(x,x) = 0$

(ii) $d(x,y) = 0 \implies x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,y) \leq d(x,z) + d(z,y)$

pseudometric : without (ii) $\implies$ example : bisimulation distances

hemimetric : without (ii) and (iii) $\implies$ example: simulation distances

quasimetric: without (iii) $\implies$ example : fuel consumption of a car

Metric space (X, d)

- Any set is a metric : $d(x,y) = \begin{cases} 0 & x = y \\ \infty & x \neq y \end{cases}$

Open-ball topology : for $x \in X$, $\varepsilon > 0$, $O_\varepsilon(x) = \{ y \in X \,/\, d(x,y) < \varepsilon \}$

A topological space is metrizable if its topology is generated by a metric

- the same topology can be metrized by different metrics

Polish space : a topological space induced by a complete separable metric space

Giry monad on $\mathbb{P}\mathrm{ol}$

Let $\mathbb{P}\mathrm{ol}$ be the full subcategory of $\mathbb{M}\mathrm{eas}$ (Borel algebras of Polish spaces)

$\mathbb{P}\mathrm{ol}$ is isomorphic to the category of Polish spaces an continuous maps

We denote by $\mathbb{P}\mathrm{ol}$ both these categories.

$$\mathbb{P}\mathrm{ol} \xrightarrow{\Delta} \mathbb{P}\mathrm{ol}$$

On objects: (X, τ) - Polish space

Take ΔX with the weakest topology making integration of bounded continuous real-valued functions continuous

$$\mu \longmapsto \int_X f \, d\mu \qquad \text{for } f : X \to \mathbb{R} \begin{cases} \text{continuous} \\ \text{bounded} \end{cases}$$

i.e., if $\lim\limits_{i \to \infty} \mu_i = \mu$, then $\lim\limits_{i\to\infty} \int f \, d\mu_i = \int f \, d\mu$

unit:

$$\eta_X : X \longrightarrow \Delta X$$

$$\eta_X(x) = \delta_x$$

multiplication:

$$\mu_X : \Delta\Delta X \longrightarrow \Delta X$$

for $\pi \in \Delta\Delta X$, $S \in Bo(X)$

$$\mu_X(\pi)(S) = \int_{\Delta X} \rho(S)\,\pi(d\rho)$$

Giry monad on Pol: (Δ, η, μ)

Theorem If X is a Polish space, then ΠX is a dense set in the weak topology on ΔX.

Q1: Is the weak topology metrizable?

Q2: What is the enriched Lawvere theory (enriched equational theory) $\mathfrak{T}$ that characterizes the Giry monad, i.e. such that

$$EM(\Delta) \cong Models(\mathfrak{T})$$

Q3: Does the algebraic structure of ΔX (Barycentric?) interact with the weak topology?

Chapter III : Kantorovich Metrics

Given a Polish space X, define a metric on ΔX

For $\mu, \nu \in \Delta X$

$$K(\mu,\nu) = \sup_{\substack{f: X \to \mathbb{R} \\ \text{nonexp.}}} \left| \int f \, d\mu - \int f \, d\nu \right|$$

$f : (X, d) \to \mathbb{R}$ nonexpansive:

$\forall x, y \in X, \quad |f(x) - f(y)| \leq d(x,y)$

From the transportation problem:

for $\mu, \nu \in \Delta X$

$$W(\mu,\nu) = \inf_{c \in \mathcal{C}(\mu,\nu)} \int_{X \times X} d(x,y)\,dc$$

Where $\mathcal{C}(\mu,\nu)$ – couplings with marginals μ and ν

$c \in \mathcal{C}(\mu,\nu)$ $\qquad c : X \times X \longrightarrow [0,1]$ – pb. distrib. s.t.

$$\pi_1 c = \mu \quad \text{and} \quad \pi_2 c = \nu$$

Kantorovich - Rubinstein Theorem:

$$K = W$$

Theorem: If X is a Polish space, then K metrizes the weak topology on ΔX.

Consequently, ΠX is dense in ΔX in the open-ball topology of K.

$\Rightarrow$ Answer to Q_1

Chapter IV : Quantitative equational reasoning

Quantitative Barycentric Algebras

Signature : $\{ +_\varepsilon \ / \ \varepsilon \in [0,1] \}$

Axioms: (B_1): $\vdash x +_1 y =_0 x$

(B_2): $\vdash x +_\varepsilon x =_0 x$

$\mathcal{K}$: (SC): $\vdash x +_\varepsilon y =_0 y +_{1-\varepsilon} x$

(SA): $\vdash (x +_\varepsilon y) +_\delta z =_0 x +_{\varepsilon\delta} \left(y +_{\frac{\delta - \varepsilon\delta}{1-\varepsilon\delta}} z \right)$

(K) : $\{ x =_\varepsilon x' , y =_\delta y' \} \vdash x +_e x' =_{e\varepsilon + (1-e)\delta} y +_e y'$

$(\Pi A, K) \models \mathcal{K}$ and $(\Delta A, K) \models \mathcal{K}$

In $\mathbb{E}\mathbf{Met}$ – extended metric spaces

$$EA(\Pi) \cong \mathrm{Models}(\mathcal{K})$$

$$\text{and } (\Pi A, K) \cong \mathrm{Free}_{\mathcal{K}}(A)$$

In $\mathbb{CS}\mathbf{Met}$ – complete separable metric spaces

$$EA(\Delta) \cong \mathrm{Models}(\mathcal{K})$$

$$\text{and } (\Delta A, K) \cong \mathrm{Free}_{\mathcal{K}}(A)$$