Introduction to Universal Coalgebra

Ezra Schoen

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Contents

Basic definitions and examples

Definition

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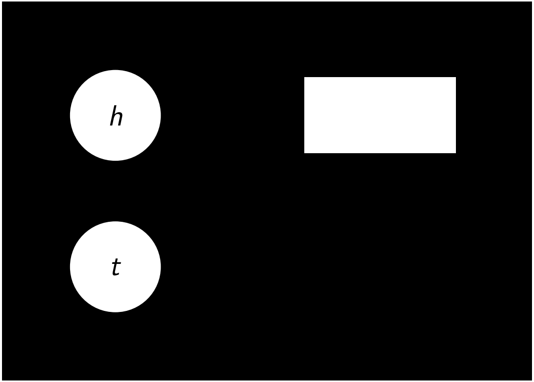

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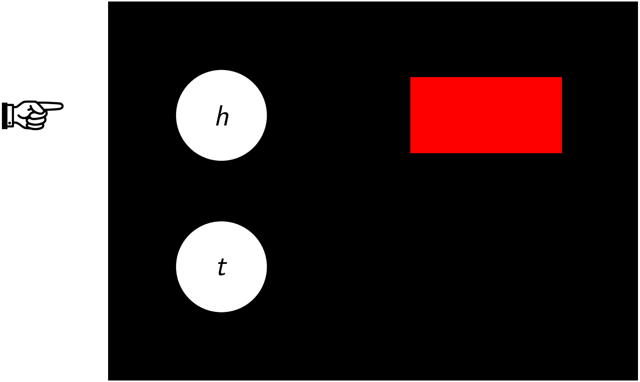

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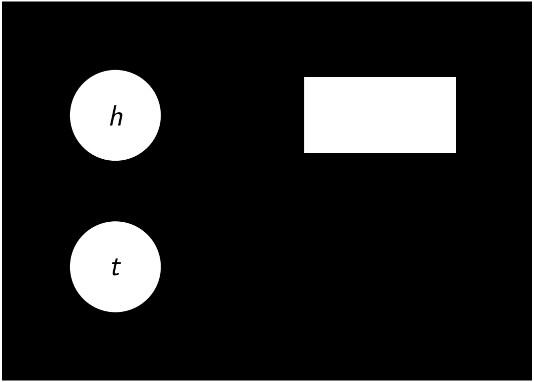

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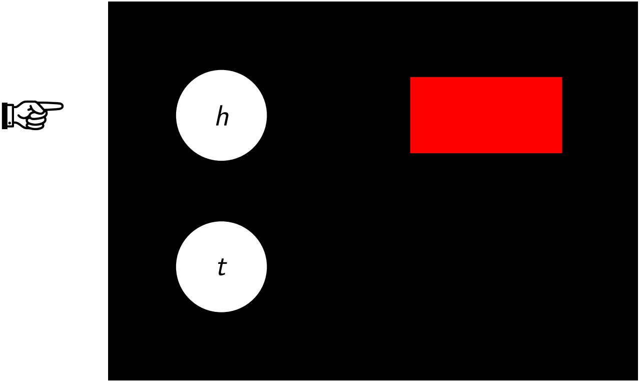

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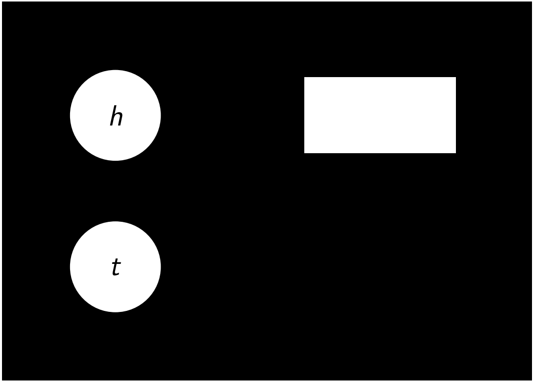

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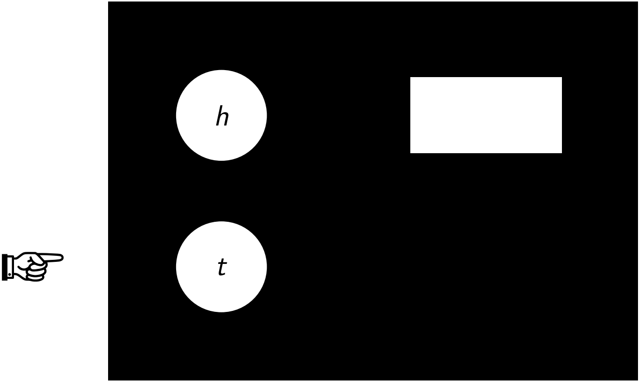

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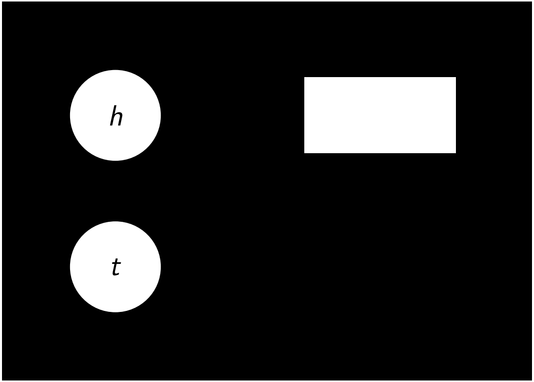

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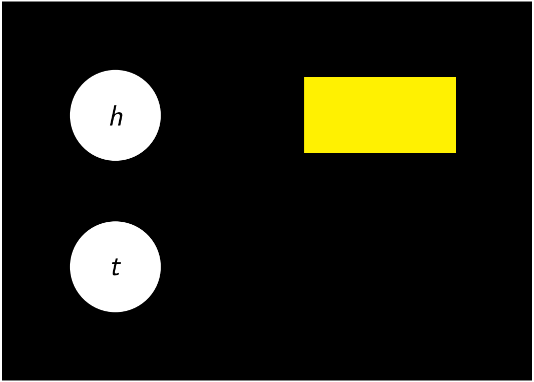

Black box machines

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$$h : S \to C, \quad t : S \to S$$

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Black box machines are $C \times \mathrm{id}$-coalgebras

Deterministic automata

A an alphabet, S a set of states.

A subset $F \subseteq S$, a function $S \times A \to S$.

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A function $\alpha : S \to \{0, 1\}$, a function $S \times A \to S$.

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A function $\alpha : S \to \{0, 1\}$, a function $\sigma : S \to S^A$.

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A deterministic automaton is a $2 \times \mathrm{id}^A$-coalgebra

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A function $\langle \alpha, \sigma \rangle : S \to \{0, 1\} \times P(S)^A$

A nondeterministic automaton is a $2 \times P(\mathrm{id})^A$-coalgebra

Final coalgebras and corecursion

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Proposition

For any BBM $\sigma = \langle h, t \rangle : S \to C \times S$, there is a *unique* coalgebra morphism $\mathrm{beh}_\sigma : S \to$ Streams.

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The only possible map is $\mathrm{beh}_\sigma(s) = (h(s), ht(s), htt(s), httt(s), \dots)$. □

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Streams is final in the category of $C \times$ id-coalgebras

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Langs is the final $2 \times \mathrm{id}^A$-coalgebra.

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interleave :: (Stream a, Stream a) -> Stream a
head $ interleave (s0, s1) = head s0
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Then there is a unique coalgebra morphism

$$\texttt{interleave} : \text{Streams} \times \text{Streams} \to \text{Streams}.$$

Behavioral equivalence and bisimulation

Behavioral equivalence

$(S, \sigma), s \simeq (S', \sigma'), s'$ iff there is a cospan

$$\begin{array}{ccc} & & (S', \sigma') \\ & & \big\downarrow{\scriptstyle g} \\ (S, \sigma) & \xrightarrow{\;f\;} & (Z, \zeta) \end{array}$$

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Behavioral equivalence is transitive via pushouts.

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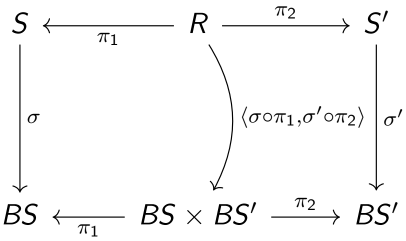

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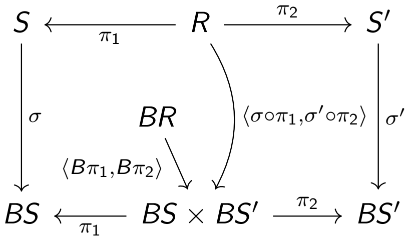

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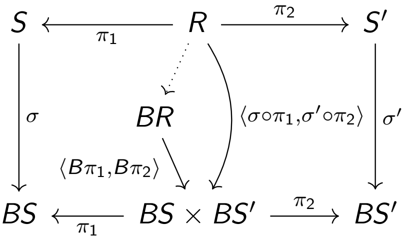

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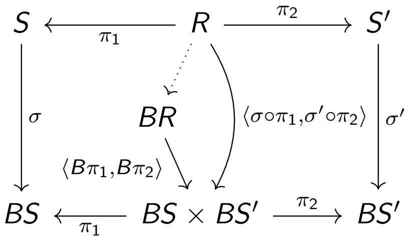

We get a span if

$$\mathrm{im}\langle \sigma \circ \pi_1, \sigma' \circ \pi_2 \rangle \subseteq \mathrm{im}\langle B\pi_1, B\pi_2 \rangle$$

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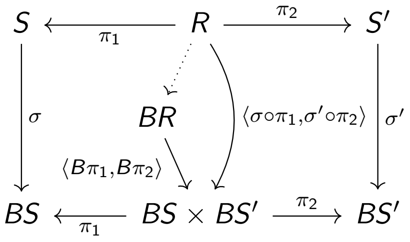

We get a span if

$$(s, s') \in R \implies \exists p \in BR : B\pi_1(p) = \sigma(s), B\pi_2(p) = \sigma'(s')$$

Bisimulation

Let B be a functor. For a relation $R : X \multimap Y$, define

$$\overline{B}R = \{(\alpha, \beta) \in BX \times BY \mid \exists p \in BR : \alpha = B\pi_1(p), \beta = B\pi_2(p)\}$$

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Bisimilar states are behaviorally equivalent. But not always the other way around!

Modal logic

Semantics of modalities

A Kripke model is a $P(\text{Prop}) \times P(-)$-coalgebra.

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Then

$$\mathfrak{M}, w \Vdash p \text{ iff } \sigma(w) \in \lambda_p(*)$$

Predicate liftings

Let B : Set $\to$ Set be a behavior functor. An *n-ary predicate lifting* is a natural transformation

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- (Labeled) binary trees: functor is $X \mapsto P(\text{Prop}) \times X \times X$. We get a binary modality $[\leftrightarrow]$ given by

$$\lambda_\leftrightarrow(U, V) = \{(A, x, y) \mid x \in U \text{ iff } y \in V\}$$

Coalgebraic modal logic

$$\mathcal{L} ::= \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \langle\lambda\rangle(\phi_1, \ldots, \phi_n)$$

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Corrollary

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- For $a \in A$, a unary lifting

$$\lambda_a(U) := \{(i,u) \mid u(a) \in U\}$$

We get a translation $m : A^* \to \mathcal{L}$ by

$$\epsilon \mapsto \langle \checkmark \rangle, \quad aw \mapsto \langle \lambda_a \rangle (m(w))$$

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Proposition

Let $\sigma : S \to BS$ be a DFA. For $s \in S$, we have that s accepts w if and only if $s \Vdash m(w)$.

Thank you for listening!

References

Bart Jacobs and Jan Rutten.
A tutorial on (co)algebras and (co)induction.
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Data.Stream

```
head :: Stream a -> a
tail :: Stream a -> Stream a

unfold :: (c -> (a,c)) -> c -> Stream a
```

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3. Related concepts

- coinduction, corecursion
- coinductive definition
- inductive type

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1. Idea

A *coinductive definition* is a definition by coinduction.

2. Definition

See at *coinductive type*.

Coinductive definitions

My conclusion

Everything is a coinductive definition