Composing Behaviors of Graphs

Sec 1: Graphs as networks
Sec 2: Open graphs
Sec 3: Compositional Dpo Sem.
Sec 4: Blackboxing

Chapter 2 of my thesis

This talk is about graphs but the math can be generalized to enriched and internal graphs. This covers a kind of wide range of networks.

A graph is a graph internal to Set. An enriched graph is a set of vertices $X$ and an object $\text{hom}(x, y)$ of $V$ for all $x, y \in X$.

Def: A graph interval to $C$ is a diagram

$E \xrightarrow{r} V$

$\frac{E}{r}$
Examples enriched set \[ \rightarrow ^* \]

- Distance Networks
- Markov Chains
- NFAs
- Connectivity (non-multi, directed)
- Capacities

Interval case

Fix a Lawvere theory \( Q \), let \( \text{Mod}(Q) \)

of models.

A graph interval to \( \text{Mod}(Q) \) is generated by a discrete net when \( Q = \text{CMON} \).

Letting \( Q = \) other stuff you get

- Leading nets, \( \star \)-safe nets
- \( \phi \)-nets...

Sec 2: Open Graphs

[Diagram]

Formally...

Defn: An open graph \( G : X \rightarrow Y \) is a cospan

\[ L X \rightarrow G \leftarrow R Y \]

Where \( L \) the discrete graph

functor \( L \)

\[ \text{Set} \xrightarrow{\phi} \text{Grp} \]

\( R(G) = \text{vertices} \), \( L(X) = \)
In any category \( \mathcal{C} \), let \( \mathcal{C} \) be the set of graphs.

Formally, let \( \mathcal{C} \) be the set of graphs.

Then, \( \mathcal{C} \) is a category, not in.

Actually, this is a category, not in.

\( x \leq d = \rho \)

\[ X \Rightarrow Y \Rightarrow Z \]

\[ G \cdot \rightarrow \approx \cdot x \to \approx \to \]

\[ y \approx \Rightarrow x \Rightarrow \approx \cdot \]

\[ L \Rightarrow G \Rightarrow L \]

\[ Z \Rightarrow \approx \Rightarrow Z \]

\[ x \approx \Rightarrow x \Rightarrow \approx \]

\[ Y \Rightarrow X \Rightarrow Y \]

\[ G \Rightarrow \approx \Rightarrow L \]

\[ x \Rightarrow x \Rightarrow \approx \]

\[ Y \Rightarrow Y \Rightarrow L \]

\[ G \Rightarrow G \Rightarrow \approx \]

\[ Y \Rightarrow Y \Rightarrow L \]
Regard $G$ as a machine. Vertices are states. Edges are events. $F(G)$ is the behavior.

Prop: There is a strict cospan in $\text{Graph}$ with objects as sets, $\text{open categories}$ up to isomorphism. We can build $F(G)$ be smaller components.

A motivating question: Can $F(G)$ be.

Defn: An open category is a cospan. $\text{Disc}$ is a pushout. Warning! This is a pushout of categories no graphs.

$F(G)$ is the operational semantics of $G$. Its under-appreciated...
Because $F$ is a left adjoint, it preserves pushouts.

Prop: There is a functor $\hat{F}: \text{Open(Graph)} \to \text{Open(cat)}$, with the identity on objects and $\hat{F}(f) = \hat{F}(G)$ for $G = FLX = DX$ and $FLY = DY$.

$\hat{F}$ preserves composition.

$\hat{F}(1 \circ G) = \hat{F}(1) \circ \hat{F}(G)$

This is pushout of graphs. This is pushout of cats.

Not useful yet. Work more simplifications.

Sec 4: Blackboxing.

Def: For an open cat $D$, its blackbox $DX \xrightarrow{c} DY$, it's blackbox is a profunctor

\[ (c) : DX \times DY \to \text{Set} \]

Defined by

\[ (c)(x, y) = \text{Hom}_C (Lx, 10y) \]

Hopefully this matches the usual thing. Is functorial?
A functor into what?

Define \( \text{Prof} \) be the 2-category where objects are categories, morphisms are profunctors, \( X \times Y^{op} \rightarrow \text{Set} \) morphisms are profunctors, \( X \times X^{op} \rightarrow \text{Set} \) composition is \( \Rightarrow \)

\[ G \circ F(x, y) = \sum_{y, z} G(x, y) \times F(y, z) \]

when \( X, Y, Z \) are discrete.

This is matrix multiplication in cardinalities - set-valued matrices.

We want \( \Rightarrow \)

\[ \text{Open}(\text{Gph}) \Rightarrow \text{Open}(\text{Cat}) \Rightarrow \text{Prof} \]

\[ \text{Prof}(\text{Gsh}) \Rightarrow \text{Prof} \]

\[ \text{Set}(\text{GstH}) \Rightarrow \text{Set}(\text{GstH}) \]

\[ \text{Set}(\text{GstH}) \Rightarrow \text{Set}(\text{GstH}) \text{ matmul} \]
Questions

Can you turn it into code?

A graph is a pair $G=(X, E)$ where $X$ is a source and every input is a sink.

$\text{Def}:
\text{A fail-free graph is a functional partial order.}$

Coalgebraic place Semantics
Questions
- Can you turn this into code?
- Coalgdaic trace
- Semantics

When is it strict?

**Def:** A *fault-open graph* is an open graph such that every input is a source and every output is a sink.

\[
\begin{array}{ccc}
\text{Source} & \rightarrow & \text{Sink} \\
\end{array}
\]

*Zeitsev functional Petri nets*
Questions
- Can you turn this into code?

Coalgebraic trace

Semantics

When is it strict?

**Definition** A **fixup-open graph** is an open graph such that every input is a source and every output is a sink.

aka Zeitser

functional Petri nets