Composing Behaviors of Graphs

Sec 1: Graphs as networks
Sec 2: Open Graphs
Sec 3: Compositional Ope Sem.
Sec 4: Blackboxing

Chapter 2 of my thesis

This talk is about graphs but the math can be generalized to enriched and internal graphs.

~~But~~ This covers a kind of wide range of networks.

Defn A graph internal to $\underline{C}$ is a diagram

$$E \underset{t}{\overset{s}{\rightrightarrows}} V$$

A graph is a graph internal to Set. An enriched graph is a set of vertices X and an object $\hom(x,y)$ of V for all $x,y \in X$

Examples

Enrich in set ↦

Enriched

- Distance Networks
- Markov Chains
- NFAs
- Connectivity (non-multi, directed)
- capacities

⋮

Interval Case

Fix a Lawvere theory Q, let $\mathrm{Mod}(Q)$ of models.

A graph interval to $\mathrm{Mod}(Q)$ is generated by a Petri net when $Q = \mathrm{CMON}$.

Letting Q = other stuff you get

- Lending nets, k-safe nets
- pre-nets ...

so WLOG...

Sec 2: Open Graphs

Formally...

Defn | An open graph $G: X \to Y$ is a cospan

$$LX \xrightarrow{i} G \xleftarrow{o} LY$$

where L the discrete graph functor

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Grph} \arrow[l, bend left, "R"] \end{tikzcd}$$

$R(G)$ = vertices $\quad L(X) = \emptyset \rightrightarrows X$

formally

$$\begin{tikzcd} & G & & H & \\ LX \arrow[ur] & & LY \arrow[ul] \arrow[ur] & & LZ \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} & & G +_{LY} H & & \\ & G \arrow[ur] & & H \arrow[ul] & \\ LX \arrow[ur] & & LY \arrow[ul] \arrow[ur] & & LZ \arrow[ul] \end{tikzcd}$$

$G : X \nrightarrow Y$ A category?

Defn) Let Open(Grph) be the cat where objects are sets, morphisms are open graphs up to iso.

$$\begin{tikzcd} & E \arrow[d, "\cong"] & \\ LX \arrow[ur] \arrow[r] & G' & LY \arrow[ul] \arrow[l] \end{tikzcd}$$

$LX = \emptyset \rightrightarrows X$

composition is pushout.

Actually this a double category... not in this talk. Structured Cospans Courser, Fong, Baez

Sec 3: Compr op. sem of graphs

Sec 3] Op Sem.

$$\begin{tikzcd} \mathrm{Grph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "U"] \end{tikzcd}$$

morphisms:
$$\begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "f"'] & V \arrow[d, "g"] \\ E' \arrow[r, shift left, "s'"] \arrow[r, shift right, "t'"'] & V' \end{tikzcd}$$

$U(C) =$ the underlying graph of C

$F(G) =$ objects are vertices of G, the morphisms are paths in G

G] It's under-appreciated...

$F(G)$ is the operational semantics of G

Regard G as a machine
vertices are states
edges are events
$F(G)$ is the behavior.

Motivating Question:
Can $F(G)$ be built from smaller components?

We can.

Defn] An open category is a cospan

$$DX \xrightarrow{i} C \xleftarrow{o} DY$$

DX, DY are the disc. on X and Y.

Prop] There is a struct. cosp cat $\mathrm{Open}(\mathrm{Cat})$ with objects as sets, morphisms are open categories up to iso

$$\begin{tikzcd} & C \arrow[d, "\cong"] & \\ DX \arrow[ur] \arrow[r] & C' & DY \arrow[ul] \arrow[l] \end{tikzcd}$$

composition is pushout.

Warning! this is a pushout of categories no graphs

~~Sec~~ Because F is a left adjoint it preserves pushouts.

Prop] There is a functor

$$\hat{F} : \mathrm{Open}(\mathrm{Graph}) \to \mathrm{Open}(\mathrm{Cat})$$

its the identity on objects

on morphisms

$$LX \xrightarrow{i} G \xleftarrow{o} LY \quad \longmapsto \quad FLX = DX \xrightarrow{Fi} FG \xleftarrow{Fo} FLY = DY$$

$\hat{F}$ preserves composition

$$\hat{F}(H \circ G) = \hat{F}(H) \circ \hat{F}(G)$$

this is pushout of graphs

this is pushout of cats

Not useful yet. We're make simplifications

Sec 4: Blackboxing.

Defn] For an open cat $DX \xrightarrow{i} C \xleftarrow{o} DY$, it's blackbox is a profunctor

$$\blacksquare(C) : DX \times DY \to \mathrm{Set}$$

defined by

$$\blacksquare(C)(x, y) = \mathrm{Hom}_C(ix, oy)$$

Hopefully this matches the usual thing. Is $\blacksquare$ functorial?

~~A~~ A functor into what?,

~~Define~~ Let Prof be the 2-category where objects are categories, morphisms are profunctors,

$$\begin{tikzcd} X \times Y^{op} \arrow[r, bend left, "A"] \arrow[d, "f \times g^{op}"'] & \mathrm{Set} \\ X' \times Y'^{op} \arrow[ur, bend right, "B"'] \arrow[u, phantom, "\Downarrow"] & \end{tikzcd}$$

are 2-morphisms.

Composition is ~~[illegible]~~

~~F~~

$$G \circ F(x, z) = \int^{y \in Y} G(x, y) \times F(y, z)$$

when X, Y, Z are discr- this is matrix multiplication in cardinalities. Set-valued matrices

We want

$$\mathrm{Open}(\mathrm{Grph}) \xrightarrow{\hat{F}} \mathrm{Open}(\mathrm{Cat}) \xrightarrow{?} \mathrm{Prof}$$

$$\hat{F}(G \circ H) \not\cong \hat{F}G \circ \hat{F}H$$

$$\hat{F}G \circ \hat{F}H \to \hat{F}(G \circ H) \text{ mult}$$

✓ matrix

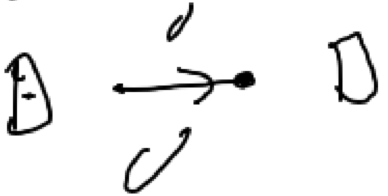
Questions

- Can you turn this into code?

- Coalgebraic trace semantics

When is it strict?

Defn A functional open graph is an open graph such that every input is a source and every output is a sink



Zaitsev
functional petri nets

Questions

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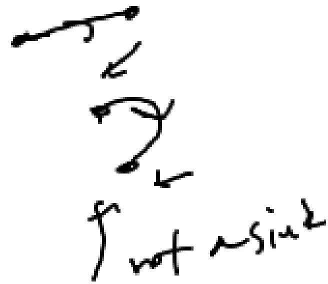
- Coalgebraic trace semantics



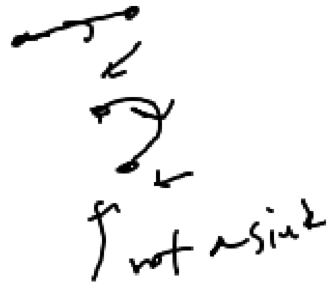
When is it strict?

Defn) A fxnl open graph is an open graph such that every input is a source and every output is a sink

Zaitsev functional petri nets

Questions

- Can you turn this into code?
- Coalgebraic trace semantics



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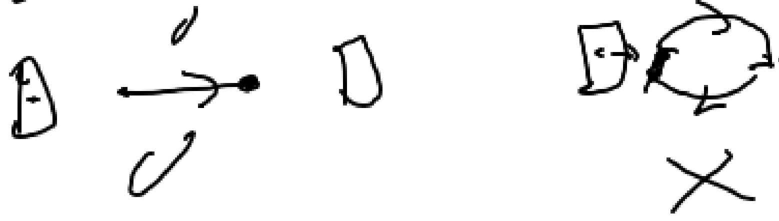

Zaitsev functional Petri nets