The Changing Shapes of Cyber Cats

MSP 101

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I am interested in
bringing formal methods
to the life sciences:

$\boxed{\text{What makes systems tick?}}$

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↳ "Active inference"...

... <u>or</u>: Functorial semantics for statistical games

Many life science models have (roughly) the following form:

1) A choice of "statistical game"

   [i.e. a (parameterized) 'Bayesian lens'

   plus a (contextual) loss / fitness function

2) A functor assigning a dynamical system to each statistical game

3) Apply (2) to (1) and simulate ...

Ex: '<u>Predictive Coding</u>'

This is a nice story, as it (roughly) seems to align with the 'modularity' observed in some neural circuits.

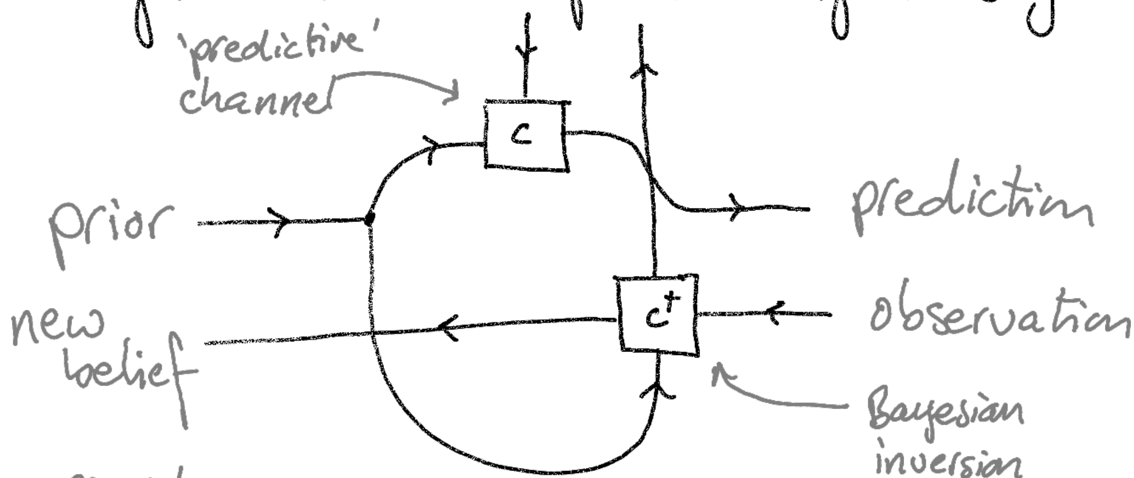
And the maths tells us why this kind of story works.

Plan

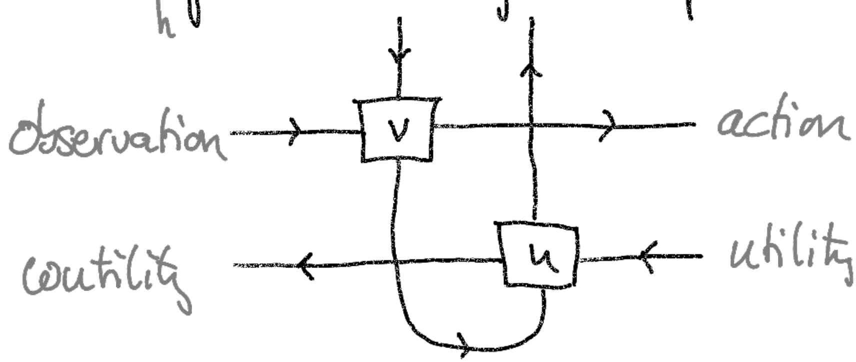
- Shapes of cyber cats
- Time in active inference
- Dynamics, categorically
- Dynamics over polynomials
- An SMC (CD-cat!) of 'generalized' coalgebras
- Open questions.

A Different Shape of Cyber Cat?

We might draw a parameterized Bayesian lens as

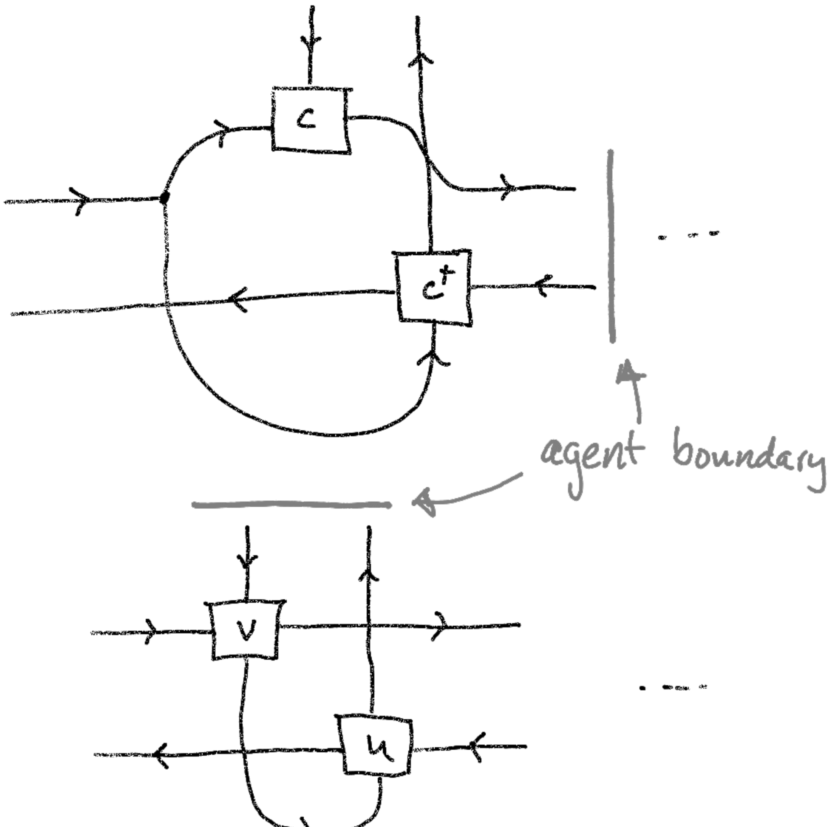


And a general parameterized optic:



But they seem to have different semantics!

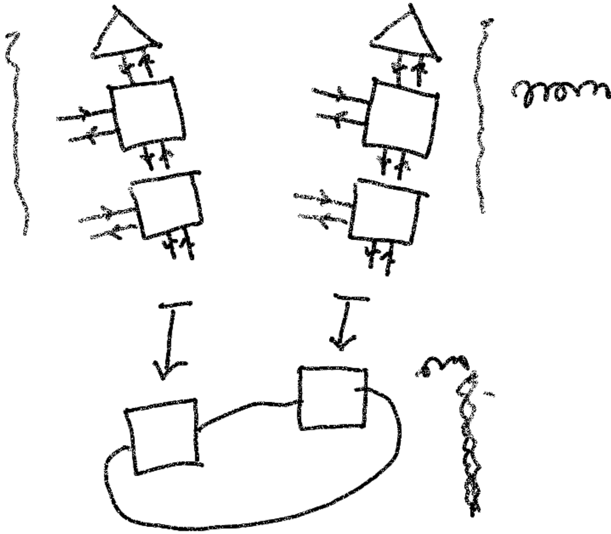
A Different Shape of Cyber Cat?



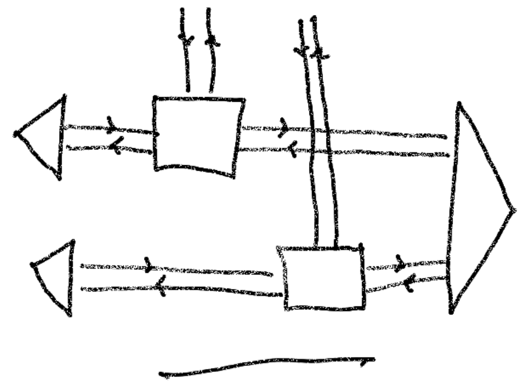
one stage in an agent's 'predictive hierarchy'

one agent - environment interaction

Two shapes of multi-agent system



vs



Two ways of thinking about time?

Time in Active Inference

1) We need an SMC for 'dynamical semantics'
   L objects: $\binom{X}{A}, \binom{Y}{B}, \ldots$
   morphisms: "(bidirectional) dynamical systems"

2) We also want to handle
   "Bayesian inference in time":
   predictions need not be 'static'

*) And somehow, these should have
   a conforming shape ...

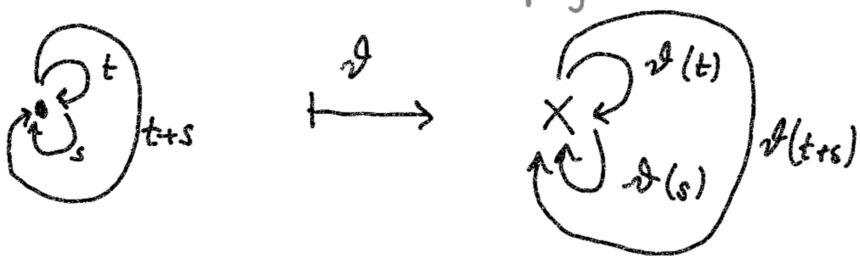
Dynamical Systems, Categorically

eg: Set, Meas, Top...

Fix a monoid T and 'category of spaces' $\mathcal{E}$.

A (closed) dynamical system is a functor $BT \longrightarrow \mathcal{E}$.

delooping



i.e. $\vartheta(s)\circ\vartheta(t) = \vartheta(t+s)$

Morphisms are natural transformations:

$$\begin{tikzcd} X \arrow[r, "\vartheta(s)"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\varphi(s)"] & Y \end{tikzcd}$$

Basic Facts

1) In discrete time, $T = \mathbb{N}$, a D.S. is just a transition map: $\vartheta(t) = \vartheta(1)^{\circ t}$

   L $\vartheta(0) = id_X$, $\vartheta(t+1) = \vartheta(t) \circ \vartheta(1)$,

   $\vartheta(2) = \vartheta(1) \circ \vartheta(1)$ ... L $\vartheta(1) : X \longrightarrow X$

2) Solutions to differential eq.^{n}s are DS.s with $T = \mathbb{R}$:

$$\frac{dx}{dt} = f(x) \implies \vartheta(t) : \begin{array}{c} x_0 \longmapsto x(t) \\ X \longrightarrow X \end{array} . \quad t : T$$

3) Can consider stochastic systems (eg Markov processes, SDEs) using functors $BT \longrightarrow \mathcal{K}\ell(P)$. probability monad

   L In discrete time, yields coalgebras $X \longrightarrow PX$ ...

What about 'open' systems? 

We can describe general open dynamical systems using "wiring diagrams" – but this does not yield an SMC:

- no 'identity' DS;
- rather, an operad-algebra.

$$A \longrightarrow \Phi \longleftarrow B$$

Alternatively: consider decorated / structured cospans.

This yields an SMC, but is only applicable for some systems – not general SDEs or other 'random dynamical systems'.

There is another way!

We can define opindexed categories

$$\mathrm{Coalg} : \mathrm{Poly}_{\mathbb{E}} \longrightarrow \mathrm{Cat}$$

of "$p^{\mathbb{P}}$-coalgebras with time $\underset{\sim}{\mathbb{T}}$"

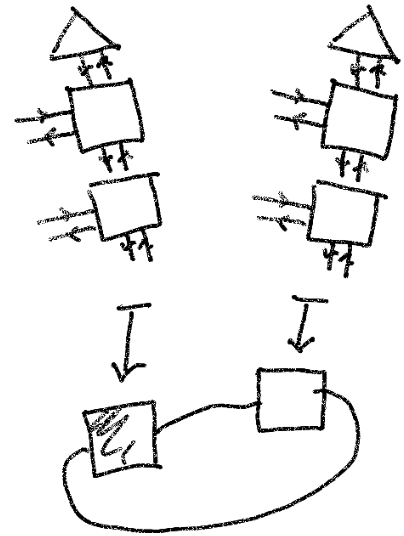
polynomial p, monad $\mathbb{P}$

and obtain an SMC by considering functors between the fibres

$$\mathrm{Coalg}(p) \longrightarrow \mathrm{Coalg}(q) \quad]$$

↳ this restricts to a copy-delete category on monomials X_y

Polynomial Functors, briefly



We will use polynomial functors to describe the 'shapes' of systems and their pattern of interaction, and use opfibrations over Poly to 'animate' them — as in the figure

Following Spivak, we will write polynomials as

$$y^A + y^B + \cdots = \sum_{i : p(1)} y^{p[i]}$$

$y^X := \mathcal{E}(-, X) : \mathcal{E} \longrightarrow \mathcal{E}$

Summands are 'configurations';
exponents are (dependent) 'incoming signals' / 'inputs'.

Polynomial Functors, briefly

$\mathbf{Poly}_\mathcal{E} \cong \int (\mathcal{E}/-)^{op}$

Grothendieck

Each polynomial $\sum_{i:p(1)} y^{p[i]}$ corresponds (by Grothendieck) to a bundle $\sum_{i:p(1)} p[i] \xrightarrow{p} p(1)$ in $\mathcal{E}$.

Morphisms $p \to q$ are pairs $(f_1, f^\#)$ as in

$$\begin{tikzcd}
\sum_{i:p(1)} p[i] \arrow[d] & \sum_{i:p(1)} q[f_1(i)] \arrow[l, "f^\#"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{j:q(1)} q[j] \arrow[d] \\
p(1) \arrow[r, equal] & p(1) \arrow[r, "f_1"] & q(1)
\end{tikzcd}$$

Composition $p \xrightarrow{(f_1, f^\#)} q \xrightarrow{(g_1, g^\#)} r$ is by Grothendieck:

$$\left(p(1) \xrightarrow{f_1} q(1) \xrightarrow{g_1} r(1),\ \sum_{i:p(1)} r[g_1 \cdot f_1(i)] \xrightarrow{f_1^* g^\#} \sum_i q[f_1(i)] \xrightarrow{f^\#} \sum_i p[i] \right)$$

Polynomial Functors, briefly

There are a number of monoidal structures on $\mathbf{Poly}_{\mathcal{E}}$.

We are concerned mainly with $(\otimes, y)$.

$$p \otimes q := \sum_{(i,j):\, p(1)\times q(1)} p[i] \times q[j]$$

NB $y := 1 \!=\!= 1$ as a bundle!

$$(f_1, f^\#) \otimes (g_1, g^\#) := (f_1 \times g_1,\ f^\# \times g^\#)$$

NB $\exists$ an embedding $\mathcal{E} \hookrightarrow \mathbf{Poly}_{\mathcal{E}} : A \longmapsto Ay$, and

$A \!=\!= A$ as bundle

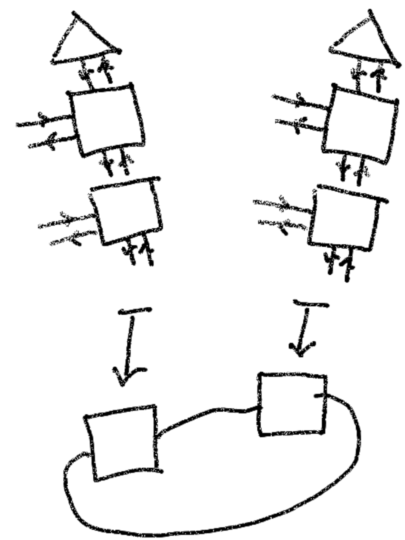
so on this subcat. of monomials,

$(\otimes, y)$ restricts to $(\times, 1)$ on $\mathcal{E}$.

Open Systems over Polynomials

Given a polynomial p, we want a category of "systems over p" that are 'open' analogs of the 'closed' systems in [BT, Kl(P)].

We can define such a category by thinking again about our figure opposite:



* We think of such systems as inhabiting a "polynomial boundary" ...

Open Systems over Polynomials

The objects of $\mathbf{Coalg}(p)$ will be triples:

(1) a state space $X : \mathcal{E}$;

equiv. a time-indexed family $\{X \longrightarrow p(1)\}_{t:T}$

(2) an output map $\vartheta^o : T \times X \longrightarrow p(1)$;

(3) an update map $\vartheta^u : \sum_{t:T} \sum_{x:X} p[\vartheta^o(t,x)] \longrightarrow \underline{pX}$

Such that, for any section $\sigma : p(1) \longrightarrow \sum_{i:p(1)} p[i]$ of p,

the 'closure' $\sum_{t:T} X \xrightarrow{\vartheta^o(t)^*\sigma} \sum_{t:T} \sum_{x:X} p[\vartheta^o(t,x)] \xrightarrow{\vartheta^u(t)} pX$

forms an object ϑ^o in $[\mathbf{B}T, \mathcal{K}\ell(p)]$.

Some Facts

1) $\mathrm{Coalg}(y) \cong \underline{[\mathcal{B}T,\ \mathcal{K}\ell(p)]}$

2) When $p = \mathrm{id}_{\mathcal{E}}$, we have deterministic systems:

$$\underline{\mathrm{Coalg}(y) \cong [\mathcal{B}T,\ \mathcal{E}]}.$$

3) When $T = \mathbb{N}$, a triple $(X, \vartheta^o, \vartheta^u)$ is equivalently a coalgebra $\vartheta : X \longrightarrow pPX$

[Such a coalgebra is equiv. typed $X \to \sum_i (PX)^{p[i]}$ which, by the universal prop. of the dependent sum, is equiv. to our pair $(\vartheta^o, \vartheta^u)$.

* So we have "pP-coalgebras with time T" !]

Coalg(p) is a category

A morphism $(X, \vartheta^0, \vartheta^1) \xrightarrow{f} (Y, \psi^0, \psi^1)$ is a map $f : X \to Y$ in $\mathcal{E}$ such that, for any section σ of p, we have a morphism $\vartheta^\sigma \to \psi^\sigma$ between the closures in $[\mathbf{B}T, \mathbb{K}\ell(P)]$:

i.e.

$$\begin{tikzcd}
X \arrow[r, "\vartheta^0(t)^*\sigma"] \arrow[d, "f"'] & \sum_{x:X} p[\vartheta^0(t,x)] \arrow[r, "\vartheta^1(t)"] & PX \arrow[d, "Pf"] \\
Y \arrow[r, "\psi^0(t)^*\sigma"] & \sum_{y:Y} p[\psi^0(t,y)] \arrow[r, "\psi^1(t)"] & PY
\end{tikzcd}$$

commutes for all $t : T$.

Coalg : $\mathbf{Poly}_2 \to \mathbf{Cat}$ is an opindexed category

Given $(X, \vartheta^o, \vartheta^u) : \mathrm{Coalg}(p)$ and $(f_1, f^\#) : p \to q$,

define $\mathrm{Coalg}(f_1, f^\#)(X, \vartheta^o, \vartheta^u)$ as the triple

$(X,\ f_1 \circ \vartheta^o,\ \vartheta^u \circ \vartheta^{o*} f^\#)$ where the two maps are explicitly

$$T \times X \xrightarrow{\vartheta^o} p(1) \xrightarrow{f_1} q(1),$$

$$\sum_{t:T} \sum_{x:X} q[f_1 \circ \vartheta^o(t,x)] \xrightarrow{\vartheta^{o*} f^\#} \sum_{t:T} \sum_{x:X} p[\vartheta^o(t,x)] \xrightarrow{\vartheta^u} PX.$$

'<u>Proof</u>': Observe that if τ is a section of q,

then $(f_1 \circ \vartheta^o(t))^* = \vartheta^o(t)^* f_1^*$ and $f^\# \circ f_1^* \tau$

is a section of p. ← use this to form the closure wrt.

An SMC of generalized pP-coalgebras

We can think of functors $\mathrm{Coalg}(p) \to \mathrm{Coalg}(q)$ as "q-shaped systems with p-shaped holes": like an 'externalization' of systems over the internal hom polynomial $[p, q]$. $\otimes$

<u>Fact</u>: $\mathrm{Coalg} : \mathbf{Poly}_\mathcal{E} \to \mathbf{Cat}$ is lax monoidal wrt. $(\otimes, y)$.

The laxator $\lambda : \mathrm{Coalg}(p) \times \mathrm{Coalg}(q) \to \mathrm{Coalg}(p \otimes q)$ takes a system over p and a system over q, and returns their product, over $p \otimes q$.

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ p(1) & & q(1) \end{tikzcd}$$

An SMC of generalized $p\mathcal{P}$-coalgebras

<u>Fact</u>: If $p = Ay^B$, $[p, y] \cong By^A$. $\quad p \to y$

<u>Fact</u>: There is an evaluation morphism $p \otimes [p, q] \to q$, which acts 'by wiring'.

Similarly, we have other 'wiring' maps: eg $Ay \otimes By^A \xrightarrow{w} By$

So, given a system with A-inputs and B-outputs — ie given a system $\vartheta : \mathrm{Coalg}(By^A)$ — we obtain a functor

$$\mathrm{Coalg}(Ay) \xrightarrow{\sim} \mathrm{Coalg}(Ay) \times 1 \xrightarrow{id \times \vartheta} \mathrm{Coalg}(Ay) \times \mathrm{Coalg}(By^A)$$

$$\xrightarrow{\lambda} \mathrm{Coalg}(Ay \otimes By^A) \xrightarrow{\mathrm{Coalg}(w)} \mathrm{Coalg}(By),$$

justifying our previous intuition.

An SMC of generalized $p\mathcal{P}$-coalgebras

We call the category of all functors between fibres of Coalg

$$\mathrm{Coalg}(\mathbf{Poly}_{\mathcal{E}}) \longrightarrow \mathbf{Cat}.$$

We write its objects as the corresponding polynomials.

Prop: A functor $F : \mathrm{Coalg}(p) \longrightarrow \mathrm{Coalg}(q)$ is equivalently a triple:

(i) an endofunctor $F_\bullet : \mathcal{E} \longrightarrow \mathcal{E}$

(ii) a T-indexed family of morphisms of polynomials

$$\{(f_1, f^*)(t) : F_\bullet p \longrightarrow q\}_{t:T}$$

(iii) a T-indexed family of nat. transformations

$$\{\phi(t) : F_\bullet P \Rightarrow PF_\bullet\}_{t:T}.$$

An SMC of generalized pP-coalgebras

Proof sketch: Look at systems over monomials Ay,

$$(S,\ \vartheta^0 : T\times S \to A,\ \vartheta^1 : T\times S \to PS).$$

A functor $F : \mathbf{Coalg}(Ay) \to \mathbf{Coalg}(By)$ must be a map on each factor of such a triple:

(i) $$\mathcal{E} \xrightarrow{F_0} \mathcal{E}$$

(ii) $$(T \to \mathcal{E}(S, A)) \longrightarrow (T \to \mathcal{E}(F_0 S, B))$$

$$\cong T \to (\mathcal{E}(-, A) \to \mathcal{E}(F_0 -, B))$$

$$\cong T \to \mathcal{E}(F_! A, B) \quad \text{by Yoneda}$$

(iii)

$$(T \to \mathcal{E}(S, PS)) \to (T \to \mathcal{E}(F_0 S, PF_0 S))$$

$$\cong T \to \{\underline{F_0 P \Rightarrow PF_0}\}$$

An SMC of generalized $p\mathcal{P}$-coalgebras

So we will henceforth work with such triples,

$$F := (F_\circ, (f_1, f^\#), \phi) : p \longrightarrow q .$$

Intuition: – $F_\circ$ returns the new state space

– $(f_1, f^\#)$ explains how to wire the system to the new polynomial

– ϕ supplies any new dynamics. ('Kl-law')

... More formally $\rightsquigarrow$

An SMC of generalized pP-coalgebras

Assume $F_\bullet$ is 'counital', $\varepsilon : F_\bullet \Rightarrow \mathrm{id}$, such that

$$F_\bullet\left(\textstyle\sum_{i:p(1)} p[i]\right) \cong \sum_{x : F_\bullet p(1)} p[\varepsilon_{p(1)}(x)] \quad \dots \quad \text{Then.}$$

"$F_\bullet$ plays well with polynomials"

Given $(F_\bullet, (f_1, f^\#), \phi) : p \to q$ and $(S, \vartheta^0, \vartheta^1) : \mathrm{Coalg}(p)$,

obtain $\Big(F_\bullet S,\ \sum_{t:T} F_\bullet S \xrightarrow{F_\bullet \vartheta^0(t)} \sum_{t:T} F_\bullet p(1) \xrightarrow{f_1(t)} q(1),$

$$\sum_{t:T} \sum_{x:F_\bullet S} q[f_1(t) \circ F_\bullet \vartheta^0(t,x)] \xrightarrow{F_\bullet \vartheta^0(t)^* f^\#(t)} \sum_{t:T} \sum_{x:F_\bullet S} p[\varepsilon_{p(1)} \cdot F_\bullet \vartheta^0(t,x)] \xrightarrow{F_\bullet \vartheta^1(t)} \sum_{t:T} F_\bullet pS \;-$$

$\xrightarrow{\phi_S(t)} pF_\bullet S\Big)$ .... and these things compose equivalently.

An SMC of generalized pP-coalgebras

Now, we want to lift $(\otimes, y)$ to $\overline{\mathrm{Coalg}(\mathbf{Poly}_c)}$.

We can just use $(\times, 1)$ on Cat, followed by the laxator: this won't be functorial!

└ We need $(F' \otimes G') \circ (F \otimes G) \cong (F'F) \otimes (G'G)$.]

But the 'naive' way duplicates the state space on the left, but not on the right.

So we need another idea ---

An SMC of generalized $p\mathcal{P}$-coalgebras

On objects, define $p \circledast q := \mathrm{Coalg}(p \otimes q)$.

Given $(F_\circ, (f_1, f^\#), \phi) : p \to p'$, $(G_\circ, (g_1, g^\#), \gamma) : q \to q'$,

define $F \circledast G : p \circledast q \to p' \circledast q'$ as the triple

$$\left(\underset{?}{F_\circ G_\circ},\ \left((f_1, f^\#) \otimes (g_1, g^\#)\right) \cdot \underset{?}{\mathrm{costr}_{FG}},\ \underline{\phi_{G_\circ} \cdot F_\circ \gamma}\right) \ldots$$

So we need some conditions, particularly:

(i) $F_\circ G_\circ \cong G_\circ F_\circ$]

(ii) $F_\circ, G_\circ$ are costrong wrt. $(\otimes, y)$, so we define

$$\mathrm{costr}_{FG} := F_\circ G_\circ (p \otimes q) \xrightarrow{(F_\circ\, \mathrm{rcostr}_G,\ \mathrm{id})} F_\circ (p \otimes G_\circ q) \xrightarrow{(\mathrm{lcostr}_F,\ \mathrm{id})} F_\circ p \otimes G_\circ q.$$

An SMC of generalized pP-coalgebras

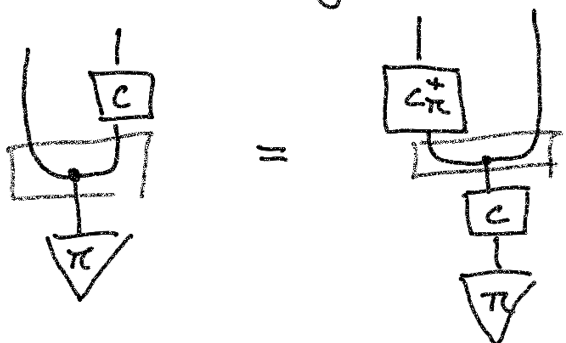
Luckily, these conditions are satisfied for the common case where we "tensor systems and wire them up", as in our example $\mathrm{Coalg}(A_y) \longrightarrow \mathrm{Coalg}(B_y)$ earlier.

* Q: How to characterize these "commutative costrong counital" functors?

(Do these conditions force them to be of the form $F = (-) \times S$?)

Copy-delete structure

We seek a notion of "Bayesian inversion in time", which means being able to draw string diagrams like



in $\overline{\mathrm{Coalg}(\mathrm{Poly}_{\mathcal{E}})}$.

Fact: There is an identity-on-objects embedding $\mathrm{Poly}_{\mathcal{E}} \hookrightarrow \overline{\mathrm{Coalg}(\mathrm{Poly}_{\mathcal{E}})}$:

$$(f_1, f^\#) \longmapsto (\mathrm{id}, \underline{(f_1, f^\#)}, \underline{\mathrm{id}}).$$

i.e. $\underbrace{\text{"just post-compose the wiring"}}$

Copy-delete structure

$X, Y, \cancel{p, q}$

We can't use the canonical comonoid structure on Cat, because again we would have 'mismatched duplications' of the state spaces — and hence no compatibility w/ the monoidal structure.

And there is no such canonical structure in $(\mathbf{Poly}_{\mathcal{E}}, \otimes, y)$, since a 'discarder' $\underline{p \to y}$ is equiv. a section of p!

But if we restrict to monomials Ay, then we can use the embeddings $\mathcal{E} \hookrightarrow \mathbf{Poly}_{\mathcal{E}} \hookrightarrow \mathbf{Coalg}(\mathbf{Poly}_{\mathcal{E}})$ and the copy-delete structure $(\Upsilon, !)$ in $(\mathcal{E}, \times, 1)$.

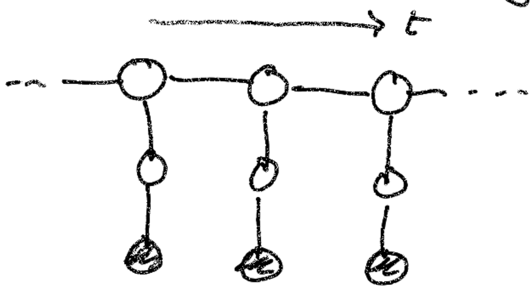
↳ We just have to restrict to $\underline{\overline{\mathbf{Coalg}}(\mathcal{E})} \hookrightarrow \overline{\mathbf{Coalg}}(\mathbf{Poly}_{\mathcal{E}})$.

Copy-delete structure + 'dynamical' statistical games

We can check that the resulting structure obeys the comonoid laws and coheres w/ the monoidal product.

We can think of this as "correlated copying", as one system now controls 2 or more 'outputs'.

We can use these gadgets to formalize Bayesian nets like

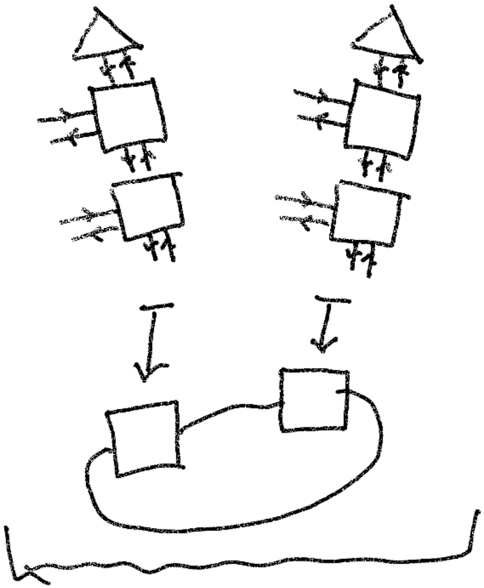


where there is both 'temporal' and 'structural' coupling ...

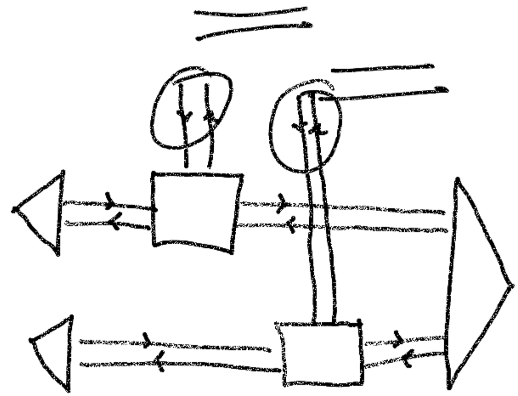
... and hence define "dynamical statistical games"!

The different shapes of cyber cats

But how to reconcile this picture? I'm still on the left!



vs



Perhaps: $\mathrm{Poly}_c \longrightarrow \mathrm{Cat}$ vs $\mathrm{Optic} \longrightarrow \mathrm{Cat}$?

What's going on here?

Nested Systems

Sometimes, we want to understand "systems w/in systems".

I've been considering systems over 1-variable polynomials:

$$1 \leftarrow E \longrightarrow B \longrightarrow 1 \qquad \longmapsfrom \qquad (S, \vartheta^{\circ}, \vartheta'')$$

But we might consider 'nested' polynomials:

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & B \arrow[d] & & (S_E, \vartheta^{\circ}, \vartheta'') \arrow[ll, mapsto] \arrow[d, mapsto] \\
J \arrow[r] & I & & (S_J, \theta^{\circ}, \theta'') \arrow[ll, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
J & E \arrow[l] \arrow[r] & B \arrow[r] & I \arrow[from=1-1, to=1-4, bend right]
\end{tikzcd}$$

We get a 'double opfibration' like this...

- What if we iterate it? (An "∞-fibration"?)
- How does this relate to 'reparameterizations"?

Coalgebraic Connections

- Are these "$p\mathcal{P}$-coalgebras in general time" well-known?
  - Do they have a logic?

    Is my Coalg(p) a topos? (I haven't checked yet...)

- Is there a neater 'coalgebraic' way of defining Coalg(p)?
  - It's easy in discrete time!

- Is $\overline{\text{Coalg}(\varepsilon)}$ traced? (I'd like it to be!)

Thanks!

I'll be adding the material on $\mathrm{Coalg}(\mathbf{Poly}_\varepsilon)$ to arXiv:2108.11137]

early next week.

→ Hancock / Setzer
"servers"
→ André ?...