Categories for persistent homology

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MSP 101 - 27 Oct 2021

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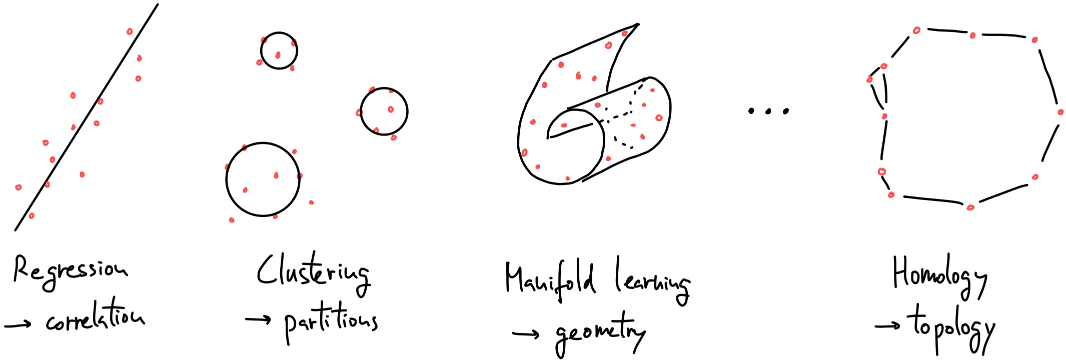
Construction

Motivating construction

Why persistent homology?

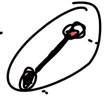


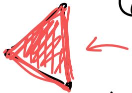
Homology of a simplicial complex

0-simplex   1-simplex   2-simplex σ

p-chain $c = \sum a_i \sigma_i$ (σ_i: p-simplex), $a_i \in \mathbb{F}$ $(\mathbb{Z}_2 = \{0,1\})$

$\partial\sigma = 1\text{-chain}$





Group of p-chains: $C_p(K, \mathbb{F})$

$$\partial : C_p(K,\mathbb{F}) \to C_{p-1}(K,\mathbb{F})$$

p-cycles $\{c \mid \partial c = 0\} = \ker \partial_p$

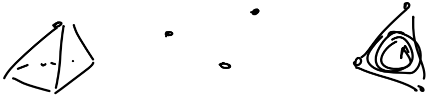
p-borders $\{c \mid c = \partial d\} = \operatorname{im} \partial_p$ (d: (p+1)-chain)

$$H_p(K,\mathbb{F}) = \frac{\text{p-cycles}}{\text{p-borders}}$$

$\to H_0(K,\mathbb{F})$ connected components

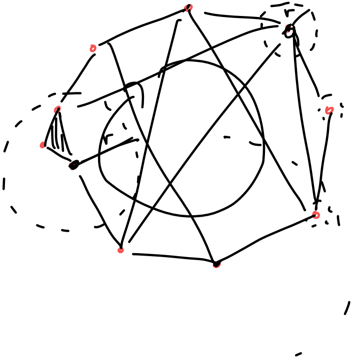
$H_1(K,\mathbb{F})$ 1-dim   holes

$H_2(K,\mathbb{F})$ 2-dim   cavities



Vietoris-Rips complex

Point cloud $\to$ simplicial complex



$$\mathcal{VR}_r(X) = \{\{u_1, u_2, \dots\} \in X \mid \|u_i - u_j\| \leq r\}$$

- $r = 0$: Everything is disconnected.
- $r \to \infty$: Everything is connected, no hole can be discerned.
- r just right: There's a hole in the middle.

Which r? Choose all!

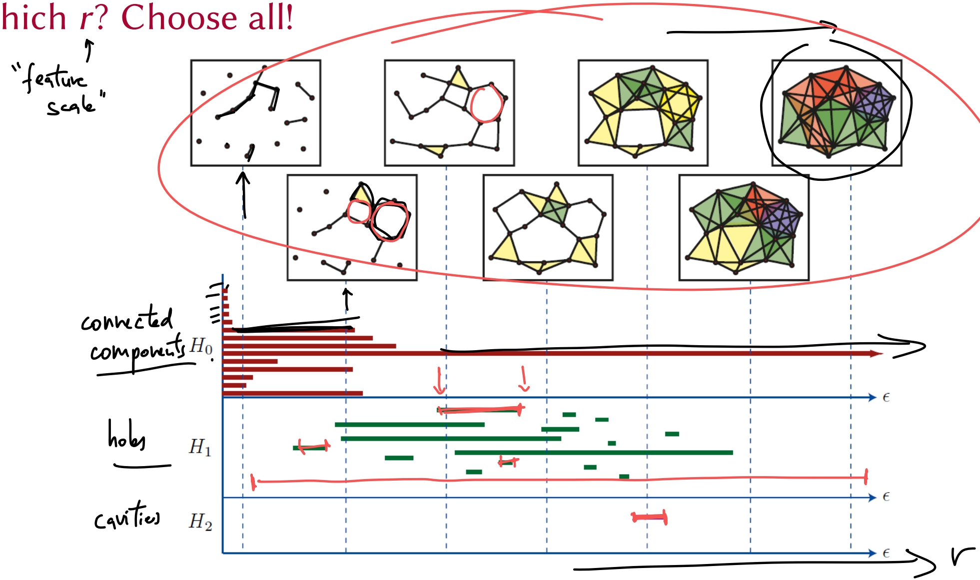
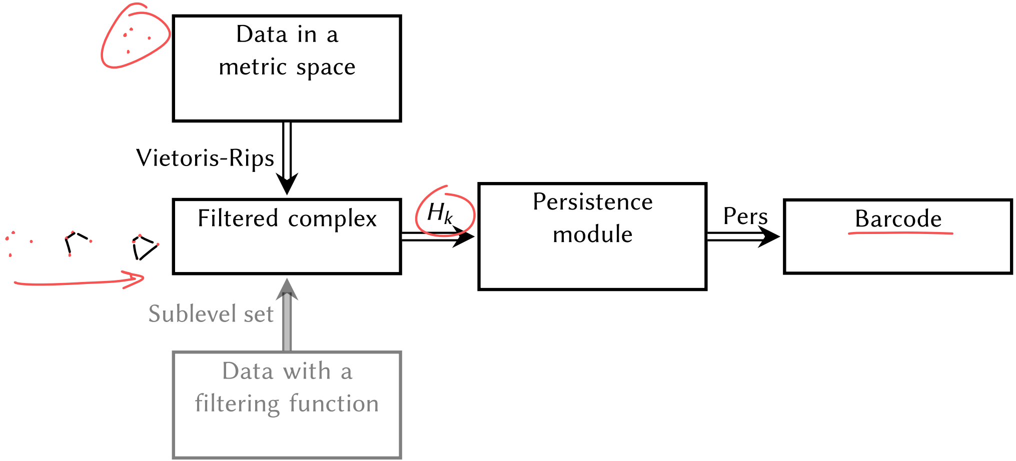


Figure 1: Barcodes for H_k [8, Fig.4]

General pipeline



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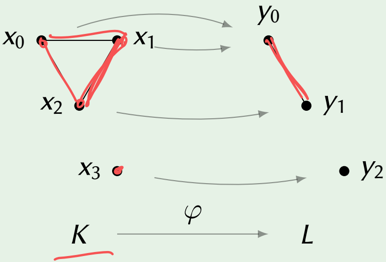
Example



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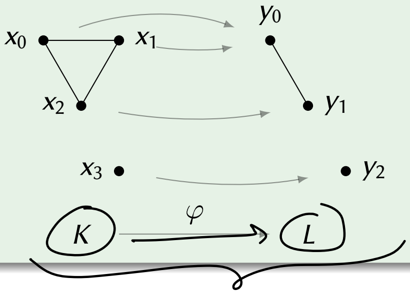
Definition ([4])

{0,1}

Simplicial k-homology (over the field $\mathbb{Z}_2$) is a functor H_k : **SCpx** $\to$ **Vec** given by $\ker(\partial_k)/\operatorname{im}(\partial_{k+1})$.

Example



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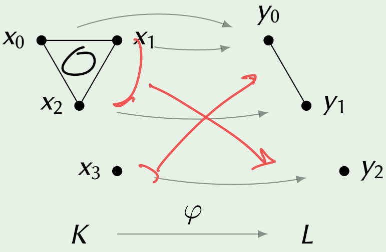

$$H_1 : \quad \mathbb{F} \xrightarrow{0} 0$$

$$H_0 : \quad \mathbb{F}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{F}^2$$

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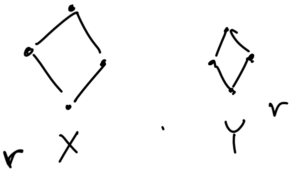
A Vietoris-Rips complex is a bifunctor

$$\mathcal{VR}\colon ([0,\infty], \geq) \times \mathbf{Met} \to \mathbf{SCpx}$$

$$(r, X) \mapsto \mathcal{VR}_r(X) = \{\{u_1, u_2, \dots\} \in X \mid \|u_i - u_j\| \leq r\}$$

$$(r \leq s, X) \mapsto \mathcal{VR}_r(X) \hookrightarrow \mathcal{VR}_s(X)$$

$$(r, X \to Y) \mapsto \mathcal{VR}_r(X) \hookrightarrow \mathcal{VR}_r(Y)$$



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$1 \to 2 \to \cdots \to n$

Persistence modules are e.g. $V_1 \to \cdots \to V_n$ in **Vec**. So they are *diagrams* indexed by $(\mathbf{n}, \leq)$

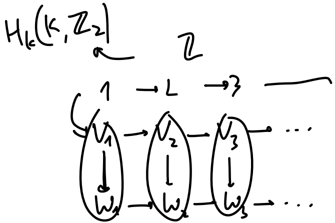
$$(\mathbf{n}, \leq) \to \mathbf{Vec}$$

A persistence module is therefore an object in $\mathbf{Vec}^{(\mathbf{n}, \leq)}$ [4].

$[(n, \leq), Vec]$

Possible generalizations of the concept

- $\mathbf{Vec}^{(\mathbb{N},\leq)}$, $\mathbf{Vec}^{(\mathbb{Z},\leq)}$, $\mathbf{Vec}^{(\mathbb{R},\leq)}$, ...
- $\mathbf{AbGrp}^{(\mathbb{N},\leq)}$, $\mathbf{Ab}^{(\mathbb{N},\leq)}$, $\mathbf{C}^{\mathbf{P}}$
- $(\mathbf{Vec^2})^{(\mathbf{n},\leq)}$ (ladder modules)
- Zigzag modules $V_1 \leftarrow V_2 \rightarrow V_3 \leftarrow V_4 \leftarrow V_5$



Decomposition

Can we decompose a persistence module into building blocks?

Persistence modules are functors $(\mathbb{N}, \leq) \to \mathbf{Vec}$.

Zigzag modules are functors:

free category

$$Rep\colon F(Q) \longrightarrow \mathbf{Vec}$$

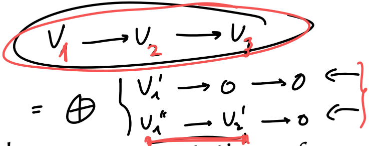
$$\bullet \to \bullet \leftarrow \bullet \to \bullet \longmapsto V_1 \to V_2 \leftarrow V_3 \to V_4$$

quiver

a representation of the quiver

They are representations of quivers (aka directed graphs).

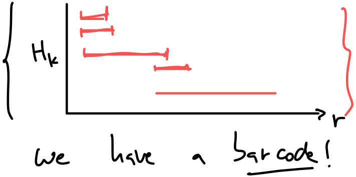
Gabriel's Theorem: Decomposition of quiver representations

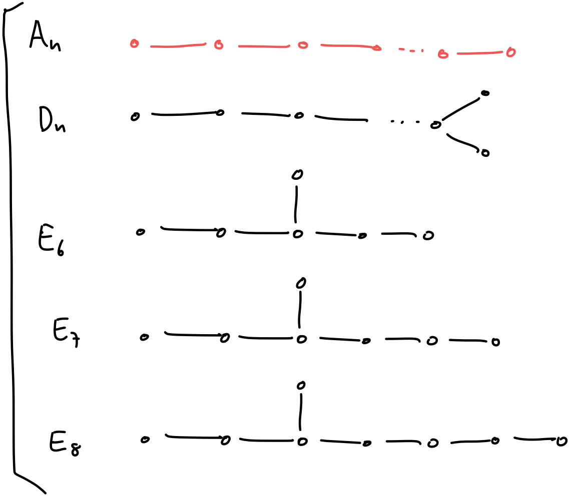


Zigzag modules are representations of quivers.

$$Rep\colon F(Q) \to \mathbf{Vec}$$

$\to$ Interval decomposition algorithm.
There are other kinds of persistence modules which are infinitely decomposable.





Comparison

Distances

We know:

- what persistence modules are
- some variations
- decompositions of certain kinds of them

How can we compare two persistence modules? Can we quantify a *distance* between topological features?

Distances

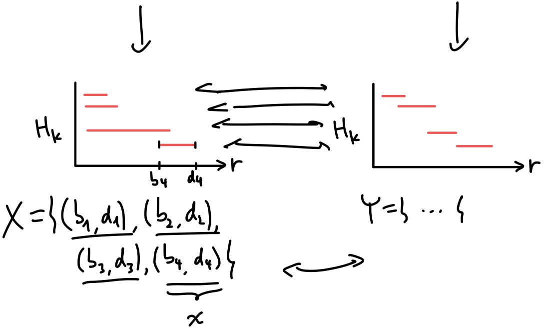
Bottleneck distance

- Compare the sets of intervals (i.e. multisets):

$$d_B(X, Y) = \inf_{\gamma\colon X \to Y} \sup_{x \in X} \|x - \gamma(x)\|_\infty$$

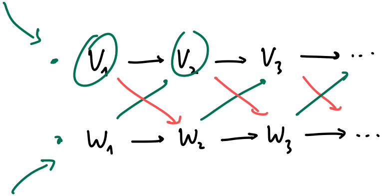
- Persistent homology is useful because the bottleneck distance is *stable*:

small changes in the point cloud $\longrightarrow$ small changes in the d_B distance



Distances

Interleaving distance

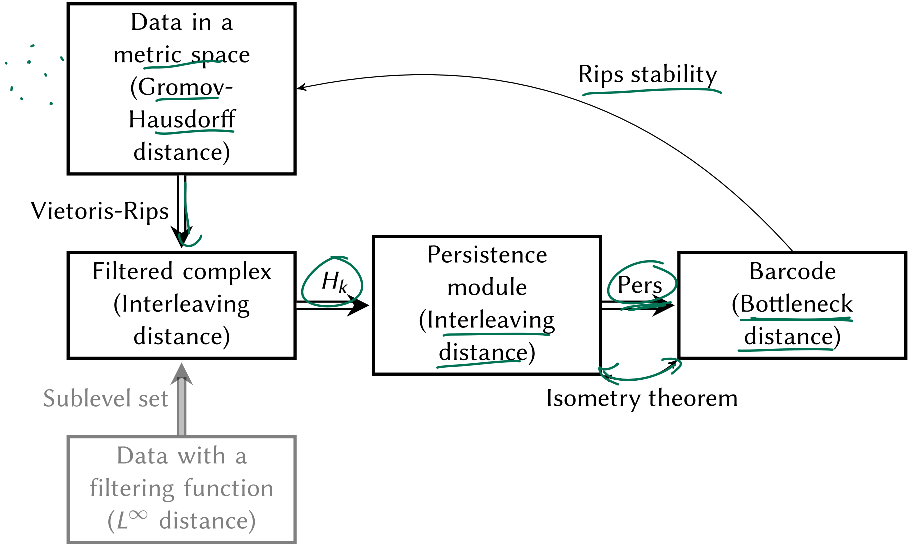


- Compare the persistence modules:
  - ε-interleaving: Pair of maps $\begin{matrix} V_i \to W_{i+\varepsilon} \\ W_i \to V_{i+\varepsilon} \end{matrix}$
  - Interleaving distance: Infimum of those ε

Theorem (Isometry theorem [5, Th.5.14])

The interleaving distance of two persistence modules in $\mathbf{Vec}^{(\mathbb{Z},\leq)}$ is the same as the bottleneck distance of their barcodes.

General pipeline, now with distances and stability



Main takeaways

- Persistence modules come from Topological Data Analysis, but have found a categorical description which has branched out and abstracted itself.
- Many theorems come from quiver theory, commutative algebra, module theory, ...
- Some kinds (but not all!) of persistence modules can be decomposed uniquely into simple building blocks.
- These can be compared quantitatively, and the distances are stable respect to the underlying point cloud metric.

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Bibliography I

[1] ASASHIBA, H., ESCOLAR, E. G., HIRAOKA, Y., AND TAKEUCHI, H.
Matrix method for persistence modules on commutative ladders of finite type.
Japan Journal of Industrial and Applied Mathematics 36, 1 (2019), 97–130.

[2] BERGOMI, M. G., FERRI, M., VERTECHI, P., AND ZUFFI, L.
Beyond topological persistence: Starting from networks.
arXiv:1901.08051 [math] (Sept. 2020).

[3] BUBENIK, P., DE SILVA, V., AND SCOTT, J.
Metrics for Generalized Persistence Modules.
Foundations of Computational Mathematics 15, 6 (2015), 1501–1531.

[4] BUBENIK, P., AND SCOTT, J. A.
Categorification of Persistent Homology.
Discrete and Computational Geometry 51, 3 (2014), 600–627.

Bibliography II

[5] Chazal, F., de Silva, V., Glisse, M., and Oudot, S.
The Structure and Stability of Persistence Modules.
SpringerBriefs in Mathematics. Springer International Publishing, Cham, 2016.

[6] Chazal, F., de Silva, V., and Oudot, S.
Persistence stability for geometric complexes.
arXiv:1207.3885 [cs, math] (Nov. 2013).

[7] de Silva, V., Morozov, D., and Vejdemo-Johansson, M.
Dualities in persistent (co)homology.
Inverse Problems 27, 12 (Dec. 2011), 124003.

[8] Ghrist, R.
Barcodes: The persistent topology of data.
Bulletin of the American Mathematical Society 45, 01 (Oct. 2007), 61–76.

Bibliography III

[9] Sheehy, D. R.
Linear-Size Approximations to the Vietoris-Rips Filtration.
arXiv:1203.6786 [cs] (Mar. 2013).

[10] Zomorodian, A., and Carlsson, G.
Computing Persistent Homology.
Discrete & Computational Geometry 33, 2 (Feb. 2005), 249–274.