Optics in three acts

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MSP101

November 17th, 2021

Act I

Tambara modules,

or

how I stopped worrying and learned to love the profunctor encoding

$\mathcal{C}, \mathcal{D}$ $\mathcal{M}$-actegories

$$\begin{array}{c} m \quad = \quad m \\ \star \qquad\qquad \star \\ a \xrightarrow{t} a' : \mathcal{C} \end{array}$$

$m \star t : m \star a \longrightarrow m \star a'$

dinat m
nat a, a'

$$\mathcal{C}(a, a') \xrightarrow{st} \mathcal{C}(m \star a, m \star a')$$

DEF. A Tambara module is $P : \mathcal{C} \nrightarrow \mathcal{D}$ equipped with

$$st_{m,a,b} : P(a,b) \longrightarrow P(m \star a, m \star b) \xrightarrow{st_n} P(n \star m \star a, n \star m \star b)$$

dinat nat

$$\begin{tikzcd} P(a,b) \arrow[rr, bend right, "st_{n \otimes m}"'] & & P(n \star m \star a, n \star m \star b) \arrow[d] \\ & & P((n \otimes m) \star a, (n \otimes m) \star b) \end{tikzcd}$$

$\mathrm{Prof}(\mathcal{C}, \mathcal{D}) \ni P : \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathrm{Set}$ 1-relations

$$a' \longrightarrow a \overset{P(a,b)}{\rightsquigarrow} b \longrightarrow b'$$

$a : \mathcal{C}$, $b : \mathcal{D}$

e.g.

$\mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$

$a \overset{\mathcal{C}(a,b)}{\rightsquigarrow} b$

$(\mathcal{C}, 1, \times) = \mathcal{D} = \mathcal{M}$

$$\begin{aligned} \mathcal{C}(a,b) &\longrightarrow \mathcal{C}(m\times a, m\times b) \\ \varphi &\longmapsto 1_m \times \varphi \end{aligned}$$

$T : \mathcal{C} \longrightarrow \mathcal{C}$ strong monad $\qquad a \times T(b) \xrightarrow{l} T(a\times b)$

$P = \mathcal{C}(-, T=) = Kl(T)(-,=)$

$$\begin{aligned} P(a,b) &\longrightarrow P(m\times a, m\times b) \\ \overset{\cup}{\varphi} &\longmapsto \end{aligned}$$

$a \to Tb$

$$\begin{tikzcd} m\times a \arrow[rr] \arrow[dr, "m\times\varphi"'] & & T(m\times b) \\ & m\times Tb \arrow[ur, "l"'] & \end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Prof}(\mathcal{C},\mathcal{D}) \arrow[r, phantom, "\perp"] \arrow[r, bend right, dashed, "\Theta"'] \arrow[r, bend left, dashed] & \mathrm{Tamb}(\mathcal{C},\mathcal{D}) \arrow[l, "U"', "\perp"]
\end{tikzcd}$$

$$P : \mathcal{C} \rightsquigarrow \mathcal{D}$$

$$P(a,b) \longrightarrow P(m*a, m*b)$$

$$f \rightsquigarrow st(f)?$$

Pastro – Street

Θ functor

$U \dashv \Theta$

$$\Theta P(a,b) = \int_m P(m*a, m*b)$$

$$\downarrow \omega$$

$$P(m*a, m*b)$$

$$P : \mathcal{C} \rightsquigarrow \mathcal{D}$$

$$P(a,b) \longrightarrow P(m \star a, m \star b)$$

P–S

$$\underline{\Psi} P(a,b) = \int^{m:\mathcal{M}} \int^{x:\mathcal{C}} \int^{y:\mathcal{D}} \mathcal{C}(a, m \star x) \times P(x,y) \times \mathcal{D}(m \star y, b)$$

$$\underline{\Psi} \dashv U \dashv \Theta$$

$U\underline{\Psi}$ monad

$$\begin{tikzcd}
\mathrm{Prof}(\mathcal{C},\mathcal{D}) \arrow[loop above] \arrow[loop below] \arrow[rr, bend left, "\underline{\Psi}"] \arrow[rr, bend right, "\Theta"'] & & \mathrm{Tamb}(\mathcal{C},\mathcal{D}) \arrow[ll, "U"']
\end{tikzcd}$$

$\perp$

ΘU comonad

$$Kl(U\Psi) \simeq coKl(\Theta U) \simeq \mathrm{Tamb}(\mathcal{C},\mathcal{D})$$

$\Psi P \Rightarrow P$ $\quad$ $P \Rightarrow \Theta P$ $\quad$ P Tamb

DEF:

$$\mathcal{O}_{\mathcal{C},\mathcal{D}}\left((\overset{\mathcal{C}}{s},\overset{\mathcal{D}}{t}),(\overset{\mathcal{C}}{a},\overset{\mathcal{D}}{b})\right) := \int_{P:\mathrm{Tamb}(\mathcal{C},\mathcal{D})} \mathrm{Set}\left(\overset{a \rightsquigarrow b}{P(a,b)},\ \overset{s \rightsquigarrow t}{P(s,t)}\right)$$

$$(x,y) \to (s,t) \to (a,b)$$

$$P(x,y) \leftarrow P(s,t) \leftarrow P(a,b)$$

Profunctor representation theorem (P–S)

$$\int_{P:\mathrm{Tamb}} \mathrm{Set}(P(a,b), P(s,t)) \cong \int^{m:M} C(s, m*a) \times D(m*b, t)$$

$$\langle v : s \to m*a,\ m : M,\ u : m*b \to t \rangle / \sim$$

<u>pf</u> $\Psi \dashv U$ + Yoneda many times

Hybrid composition

$$\begin{tikzcd}
M \arrow[d] \arrow[dr] & N \arrow[dl] \arrow[d] \\
C & D
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & m_1 n_1 m_2 n_2 \ldots \arrow[d, phantom, "\shortparallel"] & \\
M \arrow[r] & M + N \arrow[dl] \arrow[dr] & N \arrow[l] \\
C & & D
\end{tikzcd}$$

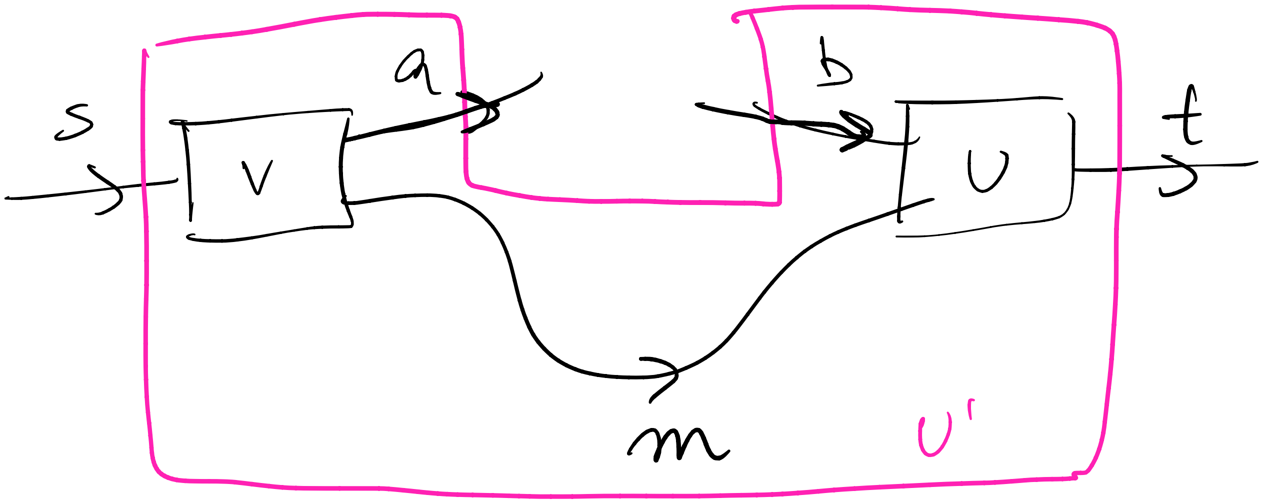
Act II

Existential optics,

or

the case for open diagrams

$$\int^{m:\mathcal{C}} \mathcal{C}(s, m * a) \times \mathcal{C}(m * b, t)$$

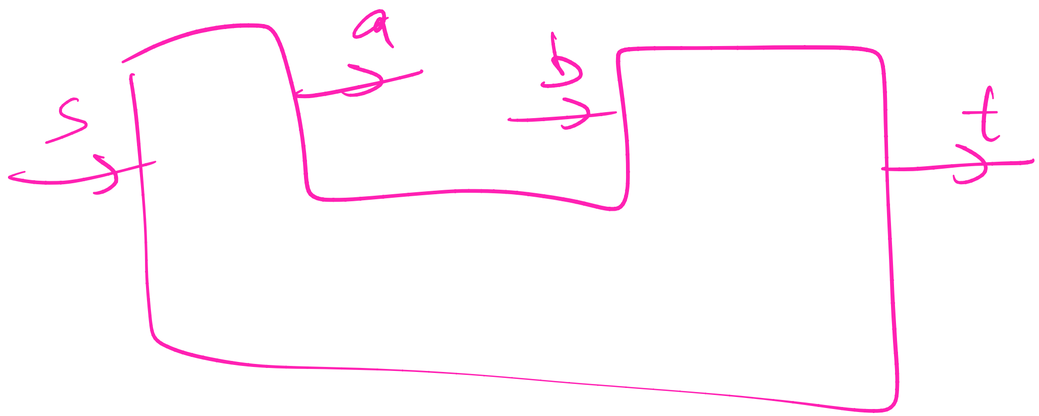


Act III

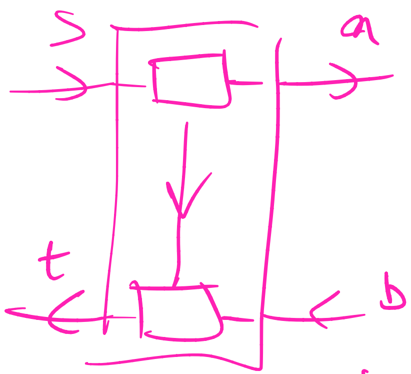
Counits,

or

how Australians turned the world upside down

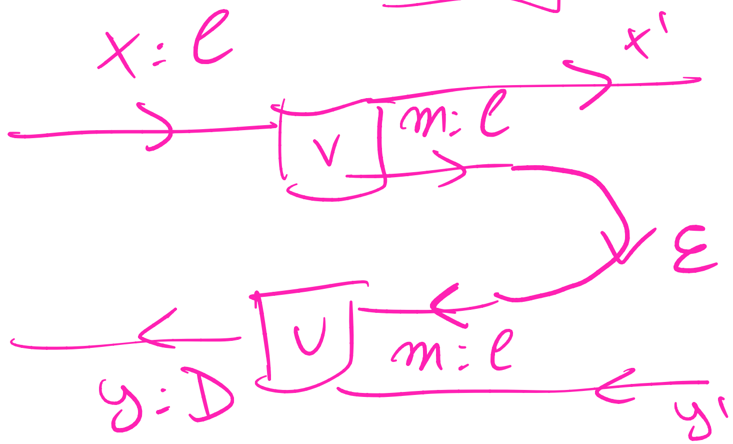


$\mathcal{C} \neq \mathcal{D} \neq \mathcal{M}$



Bicategory

Tamb



$$
\begin{array}{c}
\mathrm{Tamb}^{\mathcal{M}}(\mathcal{C}, \mathcal{D}) \times \mathrm{Tamb}^{\mathcal{M}}(\mathcal{D}, \mathcal{E}) \\
\big\downarrow j \\
\mathrm{Tamb}^{\mathcal{M}}(\mathcal{C}, \mathcal{E})
\end{array}
$$

Thanks for your attention!

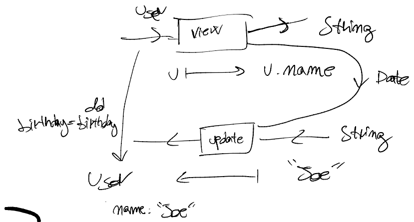
Questions?

$F : \mathcal{C} \to \mathcal{C}$

$\hookrightarrow \mathcal{C}(-, F(=)) : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Set}$

name birthday
String×Date

$(\mathrm{User}, \mathrm{User}) \xrightarrow{\text{name}} (\mathrm{String}, \mathrm{String})$



M monoid

C, D sets, M-sets

$\mathrm{Psh}(\mathcal{O})$

$C \times D \supseteq P : \mathcal{C} \times D \to 2$ $\quad \mathrm{Tamb}(\mathcal{C}, D) \cong [\mathcal{O}_{\mathcal{C},D}^{op}, \mathrm{Set}]$

$a \ni$ $\quad$ $b \ni$

$a \, R \, b$

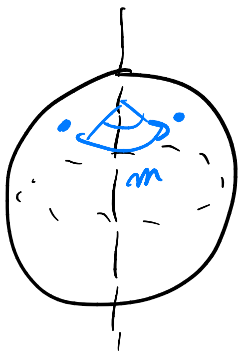
topos theory

Tambara mod $\quad$ profunctors

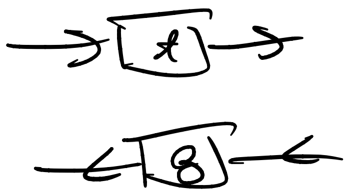
$$\begin{tikzcd} \underline{\mathrm{Psh}(\mathcal{O})} \arrow[r, ""] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \underline{\mathrm{Psh}(\mathcal{C}^{op} \times D)} \arrow[l, bend right, "\Psi"'] \arrow[l, bend left, "\Theta"] \end{tikzcd}$$

$a \, R \, b \implies m * a \, R \, m * b$

$$\mathcal{O} \longleftarrow \mathcal{C}^{op} \times D$$



$S^2 : \mathrm{Set}$ $\quad \langle f, I, g \rangle \longleftarrow\!\shortmid \ (f, g)$



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