When completeness is not enough: an introduction to algebraisable logics

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joint work with Davide Quadrellaro

Outline

Algebraisable logic is a key concept from the field of Abstract Algebraic Logic (ALL) — the general study of relations between logics and algebras.

1. Basic notion and results from AAL
2. Inquisitive logics InqB and InqI
3. Algebraising weak logics

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- 1970s — H. Rasiowa presented general theory of algebraisation for implicative logic, predecessor to AAL

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- 1970s — H. Rasiowa presented general theory of algebraisation for implicative logic, predecessor to AAL
- 1980s — W. Blok and D. Pigozzi introduced the concept of algebraisable logic. Their work is taken to be the origin of Abstract Algebraic logic.

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- Algebraisable logics
- Isomorphism theorems between deductive filters and congruences.
- Equational completeness theorems and the Tarski - Lindenbaum process
- Matrix semantics and the Leibniz congruence
- Various bridge theorems
  - Example: An finitary and finitely algebraisable logic L has the Deduction-detachment property iff its equivalent algebraic semantics has equationally definable principal relative congruences.

Logic as a consequence relation

Fix a countable set of variables $\texttt{AT}$ in a signature $\mathcal{L}$.
Let $\mathcal{F}m_{\mathcal{L}}(\texttt{AT})$ or simply $\mathcal{F}m$ be free term algebra over $\mathcal{L}$.

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A consequence relation is a relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$, s.t. for all $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq Fm$:

1. if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
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A logic of type $\mathcal{L}$ is a consequence relation $\vdash$ on the set $Fm_{\mathcal{L}}$ that is closed under uniform substitution:

3. For all substitutions σ, if $\Gamma \vdash \varphi$, then $\sigma[\Gamma] \vdash \sigma[\varphi]$.

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- Local consequence of $\mathtt{K}$:

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They have the same theorems, but $x \vdash_{\mathtt{K}_g} \Box x$, but $x \nvdash_{\mathtt{K}_l} \Box x$.

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$$\Theta \vDash_{\mathbf{Q}} \varepsilon \approx \delta \iff \text{for all } \mathcal{A} \in \mathbf{Q} \text{ and for all } h \in Hom(\mathcal{F}m, \mathcal{A}) \\ \text{if } h(x) \approx h(y) \text{ for all } x \approx y \in \Theta, \text{ then } h(\varepsilon) \approx h(\delta).$$

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The class $\mathbf{Q}$ is an algebraic semantics for a logic $\vdash$ of type $\mathcal{L}$ if there exists a set of equations $\tau(x)$, s.t.:

$$\Gamma \vdash \varphi \iff \tau[\Gamma] \vDash_{\mathbf{Q}} \tau(\varphi).$$

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- $\mathtt{K}_g \leftrightarrow$ Modal algebras, BCK logic $\leftrightarrow$ BCK algebras
- No subclass of Modal Algebras is algebraic semantics for $\mathtt{K}_l$.
- Slightly more unsettling:
  By Glivenko's theorem, $\Gamma \vdash_{\mathtt{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathtt{IPC}} \neg\neg\varphi$.
  Then it follows that $\mathtt{CPC} \leftrightarrow \mathbf{HA}$ via $\tau = \{\neg\neg x \approx 1\}$.

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The matrix $\langle \mathcal{A}, D \rangle$ is a model for a logic $\vdash$, if :

$$\begin{aligned} \Gamma \vdash \varphi \implies & \text{ for all } h \in Hom(\mathcal{F}m, \mathcal{A}), \\ & h(\gamma) \in D \text{ for all } \gamma \in \Gamma \text{ implies } h(\varphi) \in D \end{aligned}$$

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Theorem

Every logic is complete wrt to the class of its matrix models.

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1. Although both $\vDash_{\mathbf{BA}}$ and $\vDash_{\mathbf{HA}}$ interpret $\vdash_{\mathtt{CPC}}$:
   - only the **BA** interpretation can be reversed by a set of formulas $\Delta(t', t'')$:

$$\Delta(x, y) := \{x \to y, y \to x\}$$
$$\Delta(\Theta) \vdash_{\mathtt{CPC}} \Delta(\varepsilon, \beta) \iff \Theta \vDash_{\mathbf{BA}} \varepsilon \approx \delta;$$

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2. Although both $\vDash_{\mathbf{BA}}$ and $\vDash_{\{\mathbf{2}\}}$ interpret $\vdash_{\mathtt{CPC}}$, only **BA** is a class of equationally definable algebras, i.e. a variety.

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$$\begin{aligned}
\mathbb{I}(\mathbf{Q}) &:= \{\mathcal{A} : \mathcal{A} \cong \mathcal{B} \text{ for some } \mathcal{B} \in \mathbf{Q}\} \\
\mathbb{S}(\mathbf{Q}) &:= \{\mathcal{A} : \mathcal{A} \text{ is a subalgebra of some } \mathcal{B} \in \mathbf{Q}\} \\
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Theorem (Maltsev)

A class of algebras $\mathbf{Q}$ is a quasivariety if and only if it can be axiomatized by a set of quasi-equations.

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We then call $\mathbf{Q}$ the equivalent algebraic semantics for $\vdash$.

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Theorem (Uniqueness of Equivalent Semantics)

If $(\mathbf{Q}_1, \tau_1, \Delta_1)$ and $(\mathbf{Q}_2, \tau_2, \Delta_2)$ witness the algebraisability of logic $\vdash$, then:

$$(1)\ \mathbf{Q}_1 = \mathbf{Q}_2 \qquad (2)\ \tau_1(x) \mathrel{=\!\!|\!\models}_{K_i} \tau_2(x) \qquad (3)\ \Delta_1(x, y) \dashv\vdash \Delta_2(x, y).$$

Note: Proof of (3) relies on substitution invariance.

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$$\varphi := p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi$$

- for the formulas of `InqI`, we drop the classical disjunction $\varphi \vee \varphi$.

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The split property $(p \to q \mathbin{\vee\!\!\!\vee} r) \to (p \to q) \mathbin{\vee\!\!\!\vee} (p \to r)$ is a theorem of `InqI`, but replacing p with $q \mathbin{\vee\!\!\!\vee} r$ yields invalid formula.

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As a result, both `InqB` and `InqI` are not closed under uniform substitution.

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We shall extend AAL to take account for logics with weaker forms of substitution.

Weak Logics

Let Subst $:= Hom(\mathcal{F}m, \mathcal{F}m)$ and let AT be the set of substitutions σ s.t. $\sigma[\mathtt{AT}] \subseteq \mathtt{AT}$.

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Core of a Weak logic

We define the set of admissible substitutions AS of a C-logic $\vdash$ as:

$$AS(\vdash) = \{\sigma \in \mathsf{Subst} : \forall\, \Gamma \cup \{\varphi\} \subseteq Fm\; \Gamma \vdash \varphi \implies \sigma[\Gamma] \vdash \sigma(\varphi)\}.$$

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We say that $\mathcal{A}$ is equationally definable if there is some equation $\varepsilon(x) \approx \delta(x)$ such that $\text{core}(\mathcal{A}) = \{x \in \mathcal{A} : \varepsilon(x) \approx \delta(x)\}$.

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If $\mathbf{Q}$ is a class of expanded algebras and $\Theta \cup \{\varepsilon \approx \delta\}$ a set of equations, we define:

$$\begin{aligned} \Theta \vDash^c_Q \varepsilon \approx \delta \iff & \text{ for all } \mathcal{A} \in \mathbf{Q}, \\ & \text{ for all } h \in Hom(\mathcal{F}m, \mathcal{A}), \text{ s.t. } h[\mathtt{AT}] \subseteq \text{core}(\mathcal{A}) \\ & \text{ if } h(\varepsilon_i) = h(\delta_i) \text{ for all } \varepsilon_i \approx \delta_i \in \Theta, \text{ then } h(\varepsilon) = h(\delta). \end{aligned}$$

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Proposition

The core validity of a quasi-equation $\bigwedge_{i \leq n} \varepsilon_i \approx \delta_i \to \varepsilon \approx \delta$ is preserved by $\mathbb{I}, \mathbb{S}, \mathbb{P}, \mathbb{P}_U$ and $\mathbb{C}$.

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$$\Gamma \vdash \varphi \iff \tau[\Gamma] \vDash^{c}_{\mathbf{Q}} \tau(\varphi) \tag{Alg1}$$

$$\Delta[\Theta] \vdash \Delta(\eta, \delta) \iff \Theta \vDash^{c}_{\mathbf{Q}} \eta \approx \delta \tag{Alg2}$$

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$$\eta \approx \delta \;=\!\!\mid\!\vDash^c_{\mathbf{Q}} \tau[\Delta(\eta, \delta)]. \tag{Alg4}$$

We then say that $\mathbf{Q}$ is the equivalent algebraic semantics of $\vdash$.

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Let $Th^c(\mathbf{Q})$ be the set of quasi-equations true in some class of expanded algebras Q under core semantics.

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Theorem (Maltsev Theorem for Core-Generated Quasivarieties)

Let $\mathbf{Q}$ *be a quasi-variety of expanded algebras, then:*

$$\mathcal{A} \in \mathbf{Q}_{CG} \iff \mathcal{A} \vDash^c Th^c(\mathbf{Q}).$$

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The following Uniqueness Theorem then follows:

Theorem (Uniqueness of Equivalent Semantics)

If $(\mathbf{Q}_1, \tau_1, \Delta_1, \varepsilon_1 \approx \delta_1)$ *and* $(\mathbf{Q}_2, \tau_2, \Delta_2, \varepsilon_2 \approx \delta_2)$ *witness the algebraisability of a weak logic* $\vdash$, *then:*

(1) $\mathbf{Q}_1 = \mathbf{Q}_2$

(2) $\tau_1(x) \mathrel{=\joinrel\mid\models}_{\mathbf{Q}_i} \tau_2(x)$

(3) $\Delta_1(x, y) \dashv\vdash \Delta_2(x, y)$

(4) $\varepsilon_0 \approx \delta_0 \mathrel{=\joinrel\mid\models}_i \varepsilon_1 \approx \delta_1$

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- $\tau(\varphi) = \varphi \approx 1$;
- $\Delta(x, y) = x \leftrightarrow y$;
- The core is defined by $x \approx \neg\neg x$, i.e. for all $\mathcal{H} \in \mathit{Var}(\mathtt{ML})$ we have $\mathrm{core}(\mathcal{H}) = \{x \in \mathcal{H} : x = \neg\neg x\}$.

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Proof.

We let:

- $\mathbf{Q} = \mathit{Var}(\mathtt{ML})$, where $\mathtt{ML}$ is Medvedev's logic.
- $\tau(\varphi) = \varphi \approx 1$;
- $\Delta(x, y) = x \leftrightarrow y$;
- The core is defined by $x \approx \neg\neg x$, i.e. for all $\mathcal{H} \in \mathit{Var}(\mathtt{ML})$ we have $\mathrm{core}(\mathcal{H}) = \{x \in \mathcal{H} : x = \neg\neg x\}$.

Our result follows from the fact that $\mathtt{ML}$ is generated by regular elements, together with the fact that $\mathtt{InqB}$ is complete with respect to $\mathit{Var}(\mathtt{ML})$. □

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The contradiction stems from the fact that join-irreducible elements are not equationally definable. □

Properties of Algebraizable Logics

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If $\vdash$ is algebraisable with equivalent algebraic semantics $(\mathbf{Q}, \tau, \Delta, \varepsilon \approx \delta)$, then for all $\sigma \in$ Subst:

$$\sigma \in AS(\vdash) \iff \sigma \in Hom(\mathcal{F}m, \mathcal{F}m) \text{ s.t. } \sigma[\mathtt{AT}] \subseteq \mathtt{AT}.$$

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Theorem

Let $\vdash$ be algebraisable with equivalent algebraic semantics $(\mathbf{Q}, \tau, \Delta, \varepsilon \approx \delta)$, then we have $Schm(\vdash) = Log^{\tau}_{\Delta}(\mathbf{Q})$.

Duality Between Filters and Congruences for Weak Logics

Let $\vdash$ be a C-logic and $\mathcal{A}$ an expanded algebra, a set $F \subseteq \mathcal{A}$ is a deductive filter if:

$$\Gamma \vdash \varphi \implies \forall h \in Hom(Fm, \mathcal{A}), h[\mathtt{AT}] \subseteq \mathrm{core}(\mathcal{A}) \text{ and } h[\Gamma] \subseteq f \\ \text{entails } h(\varphi) \in F.$$

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Theorem

Let $\vdash$ be a weak logic with equivalent algebraic semantics $\mathbf{Q}$, then:

$$Fi_{\vdash}(\mathcal{A}) \cong Cong_{\mathbf{Q}}(\mathcal{A}) \text{ for all } \mathcal{A} \in \mathbf{Q}.$$

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What we should do next:

- Extension of our setting to non-algebraisable weak logics, e.g `InqI`.
- Applications to other logics without uniform substitution.

Thank you for your attention!

References I

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