An introduction to SGDT

+ some geometric remarks

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No non-trivial sets satisfy this!

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DT becomes a full-fledged *theory of computational spaces*!

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However, DCPOs are pretty hard to deal with. One would like to treat domains as sets $\Rightarrow$ Define a (family of) topos(es) of domains[1] $\mathcal{E}$ in which:

- One can take fixpoints of endomorphisms $X \to X$.
- One can find fixpoints for various endofunctors $\mathcal{E} \to \mathcal{E}$.
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We won't talk about that right now...

[1] All the nice topological intuition carries over in a precise way.

Suppose we want to solve:

$$\mathcal{W} \cong \mathbb{N} \rightharpoonup_{\mathsf{fin}} \mathcal{T}$$

$$\mathcal{T} \cong \mathcal{W} \to_{\mathsf{mon}} \mathcal{P}(\mathbf{V} \times \mathbf{V})$$

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Both Sets and Domains aren't that helpful with that.

Step indexing: Adding steps (natural numbers, in its most simple form) at different places in definitions in order to get a handle on recursion.

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Let's try changing a bit what we're trying to solve and take:

$$\mathcal{P}^{\downarrow}(\mathbb{N} \times \mathbf{V} \times \mathbf{V}) := \{\rho \mid (n, v, w) \in \rho \Rightarrow (m, v, w) \in \rho \ \forall m \leq n\}$$

Observe that $\mathcal{P}^{\downarrow}(\mathbb{N} \times \mathbf{V} \times \mathbf{V})$ comes equipped with the metric:

$$d(X, Y) = \inf\{2^{-n} \mid \forall j < n. \forall v, w \in \mathbf{V}.(j, v, w) \in X \leftrightarrow (j, v, w) \in Y\}$$

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$\mathcal{P}^{\downarrow}(\mathbb{N} \times \mathbf{V} \times \mathbf{V})$ is a *bisected* (1) *ultrametric* (2) space!

Let's move to the category BiCUlt of *complete bisected ultrametric spaces* and *non-expansive*[2] functions.

[2] $f : X \to Y$ is non-expansive if $\forall a, b \in X. d_Y(f(a), f(b)) \leq d_X(a, b)$.

Let's move to the category BiCUlt of *complete bisected ultrametric spaces* and *non-expansive*[2] functions.

A *locally non-expansive* functor is a BiCUlt-enriched functor.

A lne functor $F : \mathrm{BiCUlt}^{op} \times \mathrm{BiCUlt} \to \mathrm{BiCUlt}$ is *locally contractive* if $\forall f, g : X \to Y$ and $h, k : Z \to W$ we have:

$$d(F(f,h), F(g,k)) \leq \frac{1}{2} \cdot \max\{d(f,g), d(h,k)\}$$

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Remark

Composing any lne functor with the functor $\frac{1}{2} \cdot -$, which maps the space (X, d_X) to $(X, \frac{1}{2} \cdot d_X)$ and acts as the identity on morphisms, will give a locally contractive functor.

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Theorem ([BST10])
Let F an lc functor s.t. $F(1,1)$ *is inhabited. Then, there exists an inhabited* $X \in \mathrm{BiCUlt}$ *s.t.* $F(X,X) \cong X$*. If moreover* $F(\emptyset,\emptyset)$ *is inhabited, then such X is unique up to iso.*

Now we can upgrade our definition from earlier to:

$$\mathcal{T} \cong \left(\mathbb{N} \rightharpoonup_{\text{fin}} \frac{1}{2} \cdot \mathcal{T}\right) \rightarrow_{\text{mon, n.e.}} \mathcal{P}^{\downarrow}(\mathbb{N} \times \mathbf{V} \times \mathbf{V})$$

And use the fixpoint theorem for lc functors to show that it has a unique solution.

Cool! But, these spaces are awful to deal with. Again, one would like to treat such objects as sets

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Cool! But, these spaces are awful to deal with. Again, one would like to treat such objects as sets $\Rightarrow$ Define a (family of) topos(es) $\mathcal{E}$ in which:

- There's a version of Banach's fixpoint theorem[3].
- One can find fixpoints for lc endofunctors $\mathcal{E} \to \mathcal{E}$.
- There's an operator that behaves like the $\frac{1}{2} \cdot -$ functor.
- (possibly?) BiCUlt embeds into it.

[3] Contractions on a non-empty complete metric space have a unique fixpoint.

$(\mathcal{E}, \blacktriangleright\colon \mathcal{E} \to \mathcal{E}, -^{\dagger} : \mathcal{E}(\blacktriangleright -, -) \to \mathcal{E}(1, -))$, where $\mathcal{E}$ has fin. prods. and $\blacktriangleright$ is pointed[4]. s.t. (incrementally):

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- $\forall f : \blacktriangleright X \to X$, $f^{\dagger}$ is !s.t.

$$\begin{array}{ccc} 1 & \xrightarrow{f^{\dagger}} & X \\ {\scriptstyle f^{\dagger}}\downarrow & & \uparrow{\scriptstyle f} \\ X & \xrightarrow[n_X]{} & \blacktriangleright X \end{array}$$

- $\blacktriangleright$ preserves finite limits.

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- $\mathcal{E}$ is LCC + each slice is $\bullet$.

---

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Recall that an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is *strong* if $\forall X, Y. \exists F_{X,Y} : Y^X \to FY^{FX}$ s.t. $\forall f : X \to Y. F_{X,Y} \circ [\![f]\!] = [\![Ff]\!]$[5].

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A strong endofunctor on $\mathcal{E}$ is *locally contractive* if each $F_{X,Y}$ is contractive, i.e. $\exists G_{X,Y}$ s.t. $G_{X,Y} \circ n_{X^Y} = F_{X,Y}$ and the following diagrams commute:

$$\begin{array}{ccccc} \blacktriangleright(Y^X)\times\blacktriangleright(Z^Y) & \xrightarrow{\cong} & \blacktriangleright(Y^X\times Z^Y) & \xrightarrow{\blacktriangleright c} & \blacktriangleright(Z^X) \\ {\scriptstyle G_{X,Y}\times G_{Y,Z}}\downarrow & & & & \downarrow{\scriptstyle G_{X,Z}} \\ FY^{FX}\times FZ^{FY} & & \xrightarrow[c]{\qquad\qquad} & & FZ^{FX} \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\blacktriangleright [\![id]\!]} & \blacktriangleright(X^X) \\ & {\scriptstyle [\![id]\!]}\searrow & \downarrow{\scriptstyle G_{X,X}} \\ & & X \end{array}$$

[5] $[\![f]\!] : 1 \to Y^X$ is the curried version of $f : X \to Y$.

The general reference here is [BMSS11]

Proposition
If $\mathcal{E}$ is cartesian closed + $\bullet$, then $\blacktriangleright$ is strong.

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Proposition
If $\mathcal{E}$ is $\bullet$, let $F : \mathcal{E} \to \mathcal{E}$ be lc. If $X \cong F(X)$, then the two directions of the isomorphism give an initial algebra and a final coalgebra structure.

[6] $\underline{F}(\vec{X}, \vec{Y}) = \langle F(\vec{Y}, \vec{X}), F(\vec{X}, \vec{Y}) \rangle$ is the *symmetrization* of F.

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Theorem
If $\mathcal{E}$ is $\bullet$, let $F : (\mathcal{E}^{op} \times \mathcal{E})^{n+1} \to \mathcal{E}$ be lc in the (n+1)th variable pair. Then $\exists! F^\dagger : (\mathcal{E}^{op} \times \mathcal{E})^n \to \mathcal{E}$ s.t. $F \circ \langle id, \underline{F^\dagger} \rangle \cong F^\dagger$[6]. Moreover, if F is lc in all variables, then so is $F^\dagger$.

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- If $\blacktriangleright X = 1$ and $n_X = !_X$, a trivial guarded fixpoint operator is given by the identity map on the hom-sets.
- Take $\mathcal{E} = \text{BiCUlt}$, $\blacktriangleright = \frac{1}{2} \cdot -$ and n as the obvious "contracted identity" mapping. Note that a n.e. $f : \blacktriangleright X \to X$ is the same as a contractive endomap. Therefore Banach's fixpoint theorem yields a guarded fixpoint operator and BiCUlt is •.

A morphism $f : X \to Y$ is *contractive* if $\exists g : \blacktriangleright X \to Y$ s.t. $f = g \circ n_X$.
A morphism $f : X \times Y \to Z$ is *contractive in the first variable* if $\exists g$ s.t. $f = g \circ (n_X \times id_Y)$.

Theorem
All $f : X \times Y \to X$ cont. in the first variable have unique fixpoints.

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- Dulcis in fundo...

A poset A is *well-founded* if there are no infinite descending sequences $a_0 > a_1 > a_2 > \ldots$

[7] i.e. a poset with $\top$, $\bot$, all $\to$, meets and joins. Also known as *frames*.

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Let A be a poset and let $K \subseteq A$. Then K is a *basis* for A if $\forall a \in A. a = \bigvee \{k \in K \mid k \leq a\}$.

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Theorem
Let A be a complete Heyting algebra[7] *with a well-founded base. Then* $Sh(A)$ *is* •.

[7] i.e. a poset with $\top$, $\bot$, all $\to$, meets and joins. Also known as *frames*.

Let A be a well-founded poset. Then its *ideal completion* $\mathrm{Idl}(A)$ consisting of downward-closed subsets of A is a complete Heyting algebra with a well-founded basis $K = \{\downarrow a \mid a \in A\}$ where $\downarrow a = \{a' \in A \mid a' \leq a\}$.

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Proposition
If A is a poset, then $Sh(\mathrm{Idl}(A)) \simeq Psh(A)$.

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$$\mathcal{S} := Psh(\omega)$$

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Its objects are of the form:

$$X_1 \xleftarrow{r_1} X_2 \xleftarrow{r_2} X_3 \xleftarrow{r_3} \dots$$

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Its morphisms:

$$\begin{array}{ccccccc} X_1 & \xleftarrow{r_1} & X_2 & \xleftarrow{r_2} & X_3 & \xleftarrow{r_3} & \dots \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\ Y_1 & \xleftarrow[r'_1]{} & Y_2 & \xleftarrow[r'_2]{} & Y_3 & \xleftarrow[r'_3]{} & \dots \end{array}$$

The ▶ modality:

$$X \qquad X_1 \xleftarrow{r_1} X_2 \xleftarrow{r_2} X_3 \xleftarrow{r_3} X_4 \xleftarrow{r_4} \dots$$

$$\blacktriangleright X \qquad \{*\} \xleftarrow[!]{} X_1 \xleftarrow[r'_1]{} X_2 \xleftarrow[r'_2]{} X_3 \xleftarrow[r'_3]{} \dots$$

The $\blacktriangleright$ modality:

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$$\blacktriangleright X \qquad \{*\} \xleftarrow[!]{} X_1 \xleftarrow[r_1']{} X_2 \xleftarrow[r_2']{} X_3 \xleftarrow[r_3']{} \cdots$$

And its point, n_X:

$$\begin{array}{ccccccccc}
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\end{array}$$

The NNO:

$$N \qquad \mathbb{N} \xleftarrow{id_{\mathbb{N}}} \mathbb{N} \xleftarrow{id_{\mathbb{N}}} \mathbb{N} \xleftarrow{id_{\mathbb{N}}} \dots$$

The subobject classifier:

$$\Omega \qquad \{0,1\} \longleftarrow \{0,1,2\} \longleftarrow \{0,1,2,3\} \longleftarrow \dots$$

The type of streams:

$$S \cong \mathbb{N} \times S \qquad \mathbb{N}^\omega \xleftarrow{id_{\mathbb{N}^\omega}} \mathbb{N}^\omega \xleftarrow{id_{\mathbb{N}^\omega}} \mathbb{N}^\omega \xleftarrow{id_{\mathbb{N}^\omega}} \dots$$

The type of guarded streams:

$$S_\blacktriangleright \cong \mathbb{N} \times \blacktriangleright S_\blacktriangleright \qquad \mathbb{N} \xleftarrow{\pi_1} \mathbb{N}^2 \xleftarrow{\pi_{1,2}} \mathbb{N}^3 \xleftarrow{\pi_{1,2,3}} \dots$$

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Theorem
There is an equivalence between BiCUlt *and* flab($\mathcal{S}$)*, the full subcategory of flabby objects of the topos of trees.*

Proposition
A morphism in BiCUlt *is contractive in the metric sense iff it's contractive in* $\mathcal{S}$*.*

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Proposition
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There should be some geometry sneaking around!

For every topos $\mathcal{E}$, there exists a geometric morphism to Set called the *global sections* geometric morphism:

$$\Gamma : \mathcal{E} \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \text{Set} : \Delta$$

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$$\Gamma : \mathcal{E} \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} \text{Set} : \Delta$$

$$\Gamma(X) = \text{Set}(1, X) \qquad\qquad \Delta(S) = \coprod_{|S|} 1$$

A geometric morphism f is *essential* if it has an additional left adjoint $f_!$

$$\mathcal{E} \; \begin{array}{c} \xrightarrow{\quad f_! \quad} \\ \bot \\ \xleftarrow{\quad f^* \quad} \\ \bot \\ \xrightarrow[\quad f_* \quad]{} \end{array} \; \mathcal{T}$$

A geometric morphism f is *essential* if it has an additional left adjoint $f_!$

$$\mathcal{E} \underset{f_*}{\overset{f_!}{\rightleftarrows}} \mathcal{T} \qquad f_! \dashv f^* \dashv f_*$$

A topos is *locally connected* if Γ is essential.

A geometric morphism f is *essential* if it has an additional left adjoint $f_!$

$$\mathcal{E} \underset{f_*}{\overset{f_!}{\rightleftarrows}} \mathcal{T} \qquad f_! \dashv f^* \dashv f_*$$

A topos is *locally connected* if Γ is essential.

Pretty common for models of SGDT!

A geometric morphism f is *local* if it has an additional fully faithful right adjoint $f^!$

$$\mathcal{E} \underset{f^!}{\overset{f^*}{\leftleftarrows}} \mathcal{T} \qquad f^* \dashv f_* \dashv f^!, \quad f_* : \mathcal{E} \to \mathcal{T}$$

A geometric morphism f is *local* if it has an additional fully faithful right adjoint $f^!$

$$\mathcal{E} \; \begin{matrix} \xleftarrow{\qquad f^* \qquad} \\ \xrightarrow[\perp]{\quad f_* \quad \perp} \\ \xleftarrow[\qquad f^! \qquad]{} \end{matrix} \; \mathcal{T}$$

A topos is *local* if Γ is a local geometric morphism.

A geometric morphism f is *local* if it has an additional fully faithful right adjoint $f^!$

$$\mathcal{E} \underset{f^!}{\overset{f^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow[\perp]{f_* \;\; \perp}}}}} \mathcal{T}$$

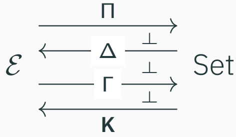
A topos is *local* if Γ is a local geometric morphism.

Is there any known local model of SGDT?

A quadruple of adjoint functors:

$$\mathcal{E} \;\; \begin{array}{c} \xrightarrow{\quad \Pi \quad} \\ \perp \\ \xleftarrow{\quad \Delta \quad} \\ \perp \\ \xrightarrow{\quad \Gamma \quad} \\ \perp \\ \xleftarrow[\quad \mathrm{K} \quad]{} \end{array} \;\; \mathrm{Set}$$

A quadruple of adjoint functors:



Exhibits the cohesion of $\mathcal{E}$ over Set if:

- $\mathbf{\Delta}$ and **K** are fully faithful.
- $\mathbf{\Pi}$ preserves finite products.

Fact: A quadruple of adjoints induces a triple of adjoints.

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There is an adjoint triple of idempotent (co)Monads on $\mathcal{E}$:

$$\mathcal{E} \underset{\Gamma}{\overset{\Pi}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\ \Delta\ }}}}} \mathrm{Set} \underset{\mathbf{K}}{\overset{\Delta}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\ \Gamma\ }}}}} \mathcal{E}$$

- The *shape* monad $\int = \Delta \circ \Pi$
- The *flat* comonad $\flat = \Delta \circ \Gamma$
- The *sharp* monad $\sharp = \mathbf{K} \circ \Gamma$

The topos $Psh(\{0 \to 1\})$[8] exhibits cohesion over Set.

[8] Also known as the *Sierpinski topos*.

The topos $Psh(\{0 \to 1\})$[8] exhibits cohesion over Set.

- Γ sends $X \to Y$ to its domain X.
- Π sends $X \to Y$ to its codomain Y.
- Δ sends a set X to the identity $X \xrightarrow{id} X$.
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The topos $Psh(\{0 \to 1\})$[8] exhibits cohesion over Set.

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Theorem
If $\mathcal{C}$ has both an initial and a terminal object, then $Psh(\mathcal{C})$ exhibits cohesion over Set *with:*

$$\lim \dashv \text{const} \dashv \text{colim} \dashv \text{coconst}$$

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In general this works for every successor ordinal

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```
-- The type of streams of naturals
-- in pseudo-haskell/idris/agda
data S : Type where
  (::) : ℕ → S → S

-- This one is recognized as productive, phew!
onOff : S
onOff = 1 :: 0 :: onOff
```

These requirements are a bit too restrictive and often confusing.

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```
interleave : S → S → S
interleave (x :: xs) ys = x :: interleave ys xs

-- Non-productive, gets rejected
dragon' : S
dragon' = interleave dragon' onOff

-- Productive, gets rejected anyways
dragon : S
dragon = interleave onOff dragon
```

Another example:

```
-- Not always non-productive
mergeBy : (ℕ → ℕ → S → S) → S → S → S
mergeBy f (x :: xs) (y :: ys) = f x y (mergeBy f xs ys)
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Can we at least save something?

Let's switch to *guarded* streams:

```
data S▸ : Type where
  (::) : ℕ → ▸S▸ → S▸

-- Remember that ▸ is an applicative
-- and has a (guarded) fixpoint operator:
fix : (▸X → X) → X
pure : X → ▸X -- a.k.a. n_X
<*> : ▸(X → Y) → ▸X → ▸Y
```

We can now fix our function:

```
mergeBy : (ℕ → ℕ → ▶S▶ → S▶) → S▶ → S▶ → S▶
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Cool! But something's off...

Adding ▶ alone is too rigid for productivity, for example:

```
dropSnd (x :: y :: xs) = x :: dropSnd xs
```

Violates causality[9], and cannot be typed using $S_{\blacktriangleright}$.

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Possible solutions: [AM13][10] [BBM14] [Gua18]

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Fortunately, we have the right "modality" for our problem:

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With the fortunate consequence:

$$\flat S_{\blacktriangleright} \cong \flat(\mathbb{N} \times \blacktriangleright S_{\blacktriangleright}) \cong \flat \mathbb{N} \times \flat(\blacktriangleright S_{\blacktriangleright}) \cong \mathbb{N} \times \flat S_{\blacktriangleright} \cong S$$

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Sadly, $\flat$ is not a type constructor.

Formal fact: In a topos, all the idempotent comonads fibred over the codomain fibration are of the form $\Box_U(A) = A \times U$ for a subterminal object U.

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Formal fact: In a topos, all the idempotent comonads fibred over the codomain fibration are of the form $\Box_U(A) = A \times U$ for a subterminal object U.

Clearly, $\flat$ doesn't have this form $\Rightarrow$ We cannot have $\flat$ as an operation Type $\to$ Type.

Possible solution: In presence of a $\sharp$ modality, we can describe $\flat$ as an operation $\sharp\mathrm{Type} \to \sharp\mathrm{Type}$.

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- Easily formalizable in an existing proof assistant [Shu11].

Cons:

- Requires a lot of work on the theory of $\sharp\mathrm{Type}$.
- It's hard to "escape" from $\sharp\mathrm{Type}$ [Shu11, Shu18].

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- SGDT is both a generalization of step indexing in categories of metric spaces [BST10] and Nakano-style guarded recursion [Nak00].

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- SGDT is both a generalization of step indexing in categories of metric spaces [BST10] and Nakano-style guarded recursion [Nak00].
- Models of SGDT are pretty easy to come by and (often) have simple descriptions in terms of very simple presheaf categories.
- Although there are many signs of geometry hiding in plain sight in SGDT this aspect of the theory has been pretty much ignored as of now.
- The $\blacktriangleright$ modality saves us from coding around syntactic productivity checks but it’s too rigid when considered alone, $\flat$ and $\sharp$ help us with that.

See you ▶, s!

- Robert Atkey and Conor McBride, Productive coprogramming with guarded recursion, ACM SIGPLAN Notices **48** (2013), no. 9, 197–208.
- Aleš Bizjak, Lars Birkedal, and Marino Miculan, A model of countable nondeterminism in guarded type theory, Rewriting and Typed Lambda Calculi, Springer, 2014, pp. 108–123.
- Lars Birkedal, Rasmus Ejlers Mogelberg, Jan Schwinghammer, and Kristian Stovring, First steps in synthetic guarded domain theory: step-indexing in the topos of trees, 2011 IEEE 26th Annual Symposium on Logic in Computer Science, IEEE, 2011, pp. 55–64.

Lars Birkedal, Kristian Støvring, and Jacob Thamsborg, The category-theoretic solution of recursive metric-space equations, Theoretical Computer Science **411** (2010), no. 47, 4102–4122.

Adrien Guatto, A generalized modality for recursion, Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, 2018, pp. 482–491.

Hiroshi Nakano, A modality for recursion, Proceedings Fifteenth Annual IEEE Symposium on Logic in Computer Science (Cat. No. 99CB36332), IEEE, 2000, pp. 255–266.

Jonathan Sterling and Robert Harper, Guarded computational type theory, Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, 2018, pp. 879–888.

Michael Shulman, Internalizing the external, or the joys of codiscreteness, 2011, Blog Post available at `https://golem.ph.utexas.edu/category/2011/11/internalizing_the_external_or.html`.

______, Brouwer's fixed-point theorem in real-cohesive homotopy type theory, Mathematical Structures in Computer Science **28** (2018), no. 6, 856–941.