MSP 101

Introduction to Game Semantics

Jérémy Ledent

Thursday 10 February 2022

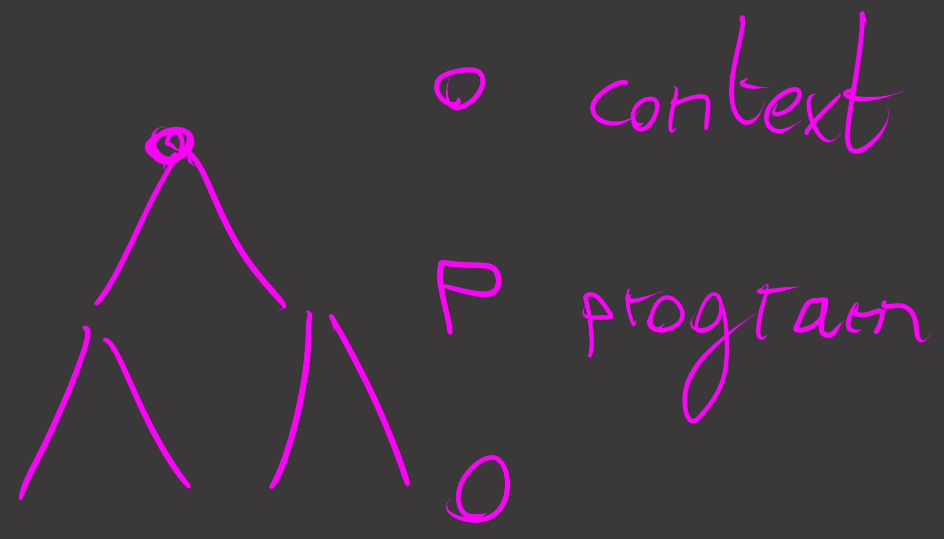
Brief overview

- Game Semantics originated in the 90's and comes in two flavors
  - ↳ AJM - style games (Abramsky, Jagadeesan Malacaria)
  - ↳ HO - style games (Hyland, Ong)
- Solves the full abstraction problem for PCF

  λ-calculus
  +Nat + Bool +Y
- Idea:

  Types ⟷ Arenas

  Programs ⟷ P-Strategies



Syntax, semantics

- Syntax: programs are chains of characters

  ex: $\lambda f. \lambda x. f\ x$

  ex:
  ```
  eval :: (a -> b) -> a -> b
  eval f x = f x
  ```

+ Grammar rules
+ Typing rules

Syntax, semantics

- Syntax: programs are chains of characters

<u>ex:</u> $\lambda f. \lambda x. f\ x$

<u>ex:</u>

```
eval :: (a -> b) -> a -> b
eval f x = f x
```

+ Grammar rules
+ Typing rules

- Operational semantics: programs compute
  - * Small step: $(\lambda x. t)\ u \longrightarrow_{\beta} t[{}^{u}/_{x}]$
  - * big step: $\text{eval}\ (\backslash x \to x + x)\ 7\ \boxed{\Downarrow 14}$

    $\text{let } x = x \text{ in } x\ \boxed{↯}$

Syntax, semantics

- Syntax: programs are chains of characters

  ex: $\lambda f. \lambda x. f\ x$

  ex:

  ```
  eval :: (a -> b) -> a -> b
  eval f x = f x
  ```

\+ Grammar rules

\+ Typing rules

- Operational semantics: programs compute
  - Small step: $(\lambda x. t)\ u \longrightarrow_\beta t[u/x]$
  - big step: `eval (\x -> x+x) 7` $\Downarrow 14$

    `let x = x in x` ↯

- Denotational semantics: interpret a program p mathematical object $\llbracket p \rrbracket$.

  ex:

  ```
  twice :: Integer -> Integer
  twice x = x + x
  ```

  $\rightsquigarrow$ $\llbracket \text{twice} \rrbracket : \mathbb{Z} \to \mathbb{Z}$

  $x \mapsto 2x$

Denotational semantics for λ-calculus

- Interpret the types:

$[\![Nat]\!] = ?$ $\quad$ $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$

$[\![Bool]\!] = ?$ $\quad$ $[\![A \to B]\!] = [\![A]\!] \Rightarrow [\![B]\!]$

- Interpret the typing contexts:

$$\Gamma = x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$$

$$[\![\Gamma]\!] = [\![A_1]\!] \times [\![A_2]\!] \times \ldots \times [\![A_n]\!]$$

- Interpret the well-typed terms:

if $\Gamma \vdash t : A$,

define $[\![t]\!] : [\![\Gamma]\!] \longrightarrow [\![A]\!]$

↳ Cartesian-closed category

Examples:

- Sets and (partial) functions
- CPO$_\perp$ and continuous functions

Properties of a "good" denotational semantics

- Soundness:

  if $t \rightarrow_\beta u$, then $[\![t]\!] = [\![u]\!]$

- Compositionality:

  for every context $C[-]$,

  if $[\![t]\!] = [\![u]\!]$, then $[\![C[t]]\!] = [\![C[u]]\!]$

- Definability:

  for every morphism $h : [\![\Gamma]\!] \longrightarrow [\![A]\!]$

  there exists $\Gamma \vdash t : A$ such that $[\![t]\!] = h$

Properties of a "good" denotational semantics

- Soundness:

  if $t \to_\beta u$, then $[\![t]\!] = [\![u]\!]$

- Compositionality:

  for every context $C[-]$,

  if $[\![t]\!] = [\![u]\!]$, then $[\![C[t]]\!] = [\![C[u]]\!]$

- Definability:

  for every morphism $h : [\![\Gamma]\!] \longrightarrow [\![A]\!]$

  there exists $\Gamma \vdash t : A$ such that $[\![t]\!] = h$

$t \equiv_{obs} u$

Def: two programs t, u are observationally equivalent when for every context $C[-]$ s.t. $C[t]$ and $C[u]$ are programs, $C[t] \Downarrow v$ iff $C[u] \Downarrow v$

Properties of a "good" denotational semantics

- Soundness:

  if $t \to_\beta u$, then $[\![t]\!] = [\![u]\!]$

- Compositionality:

  for every context $C[-]$,

  if $[\![t]\!] = [\![u]\!]$, then $[\![C[t]]\!] = [\![C[u]]\!]$

- Definability:

  for every morphism $h : [\![\Gamma]\!] \longrightarrow [\![A]\!]$

  there exists $\Gamma \vdash t : A$ such that $[\![t]\!] = h$

Def: two programs t, u are observationally equivalent when for every context $C[-]$ s.t. $C[t]$ and $C[u]$ are well-typed, $C[t] \Downarrow v$ iff $C[u] \Downarrow v$

- Adequacy: $[\![t]\!] = [\![u]\!] \implies t \equiv_{obs} u$

- Full Abstraction:

  Scott $\boxed{\text{PCF}}$ + par-or

  $[\![t]\!] = [\![u]\!]$ iff $t \equiv_{obs} u$

Game

Semantics

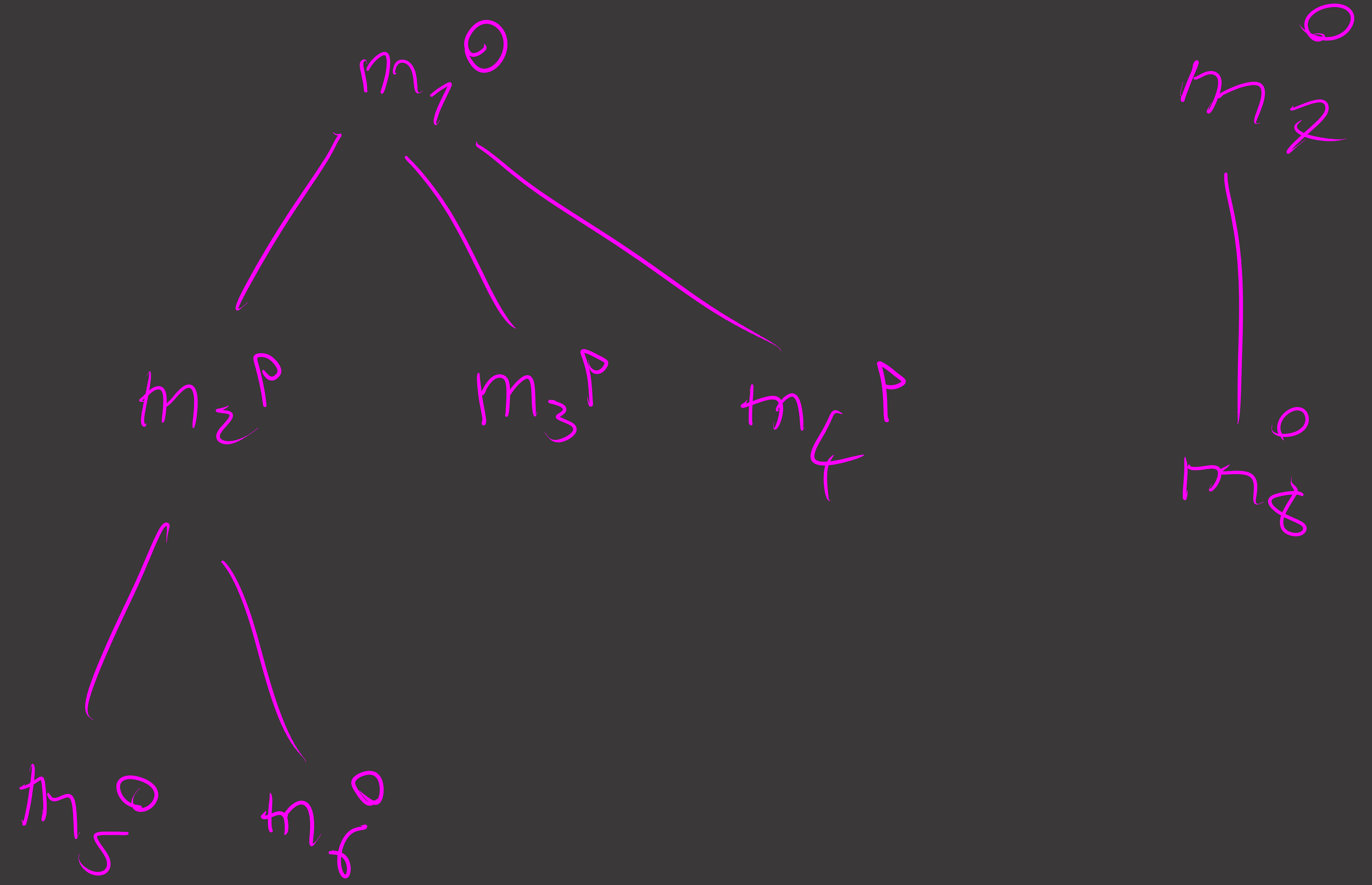
Arenas

Def: An arena $A = \langle M, \vdash, \lambda \rangle$ is given by:

- A set of moves M,
- An enabling relation $\vdash \subseteq (M \cup \{*\}) \times M$,
- A polarity function $\lambda : M \longrightarrow \{O, P\}$

Such that:

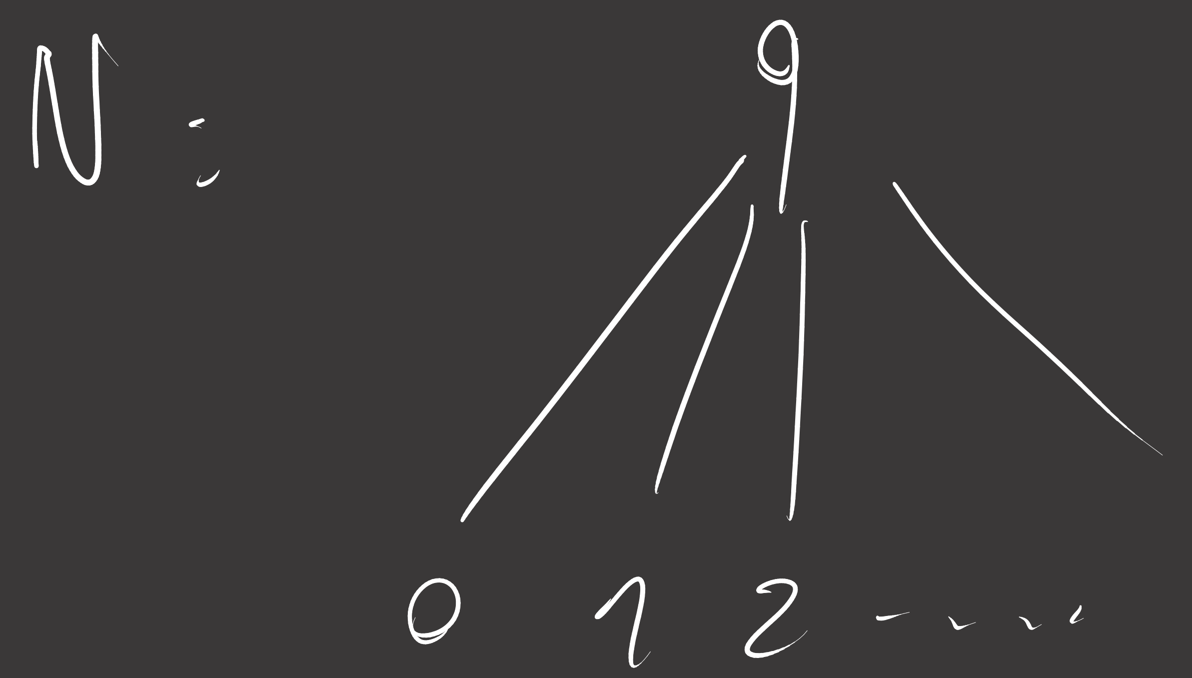
- if $* \vdash m$ then $\lambda(m) = O$
- if $m \vdash n$ then $\lambda(m) \neq \lambda(n)$

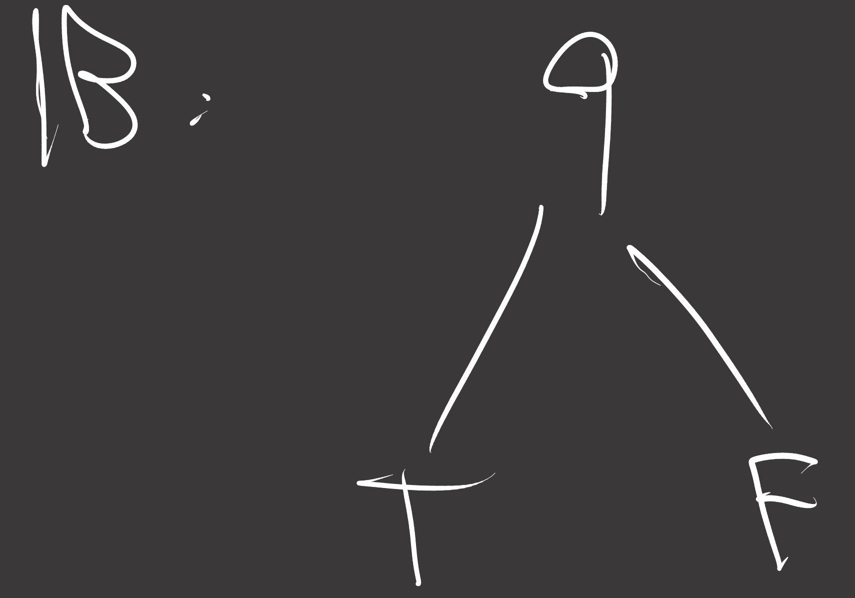


Operations on arenas

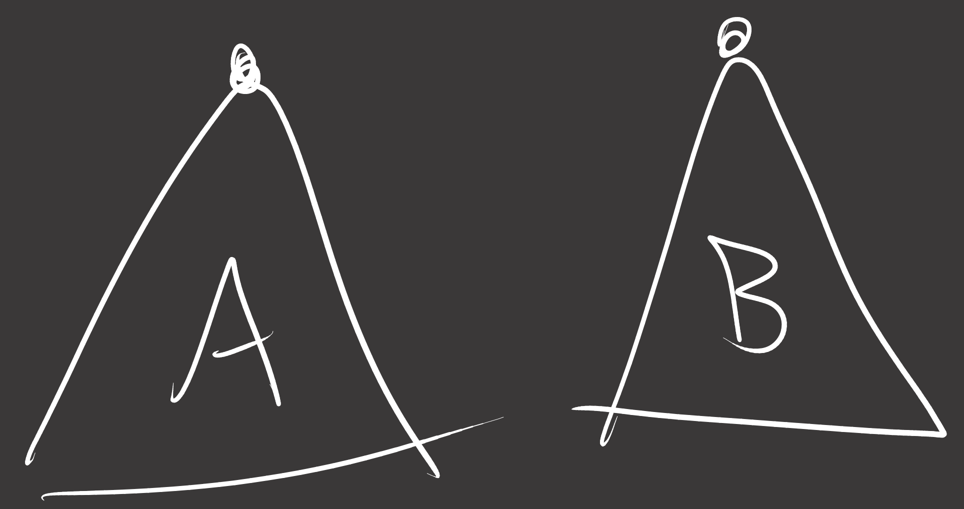
$[\![\text{Nat}]\!] = N$

$[\![\text{Bool}]\!] = B$

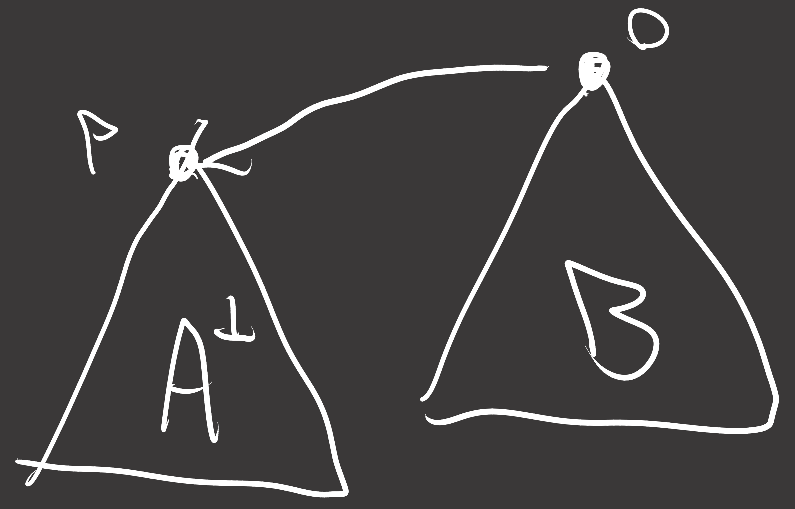




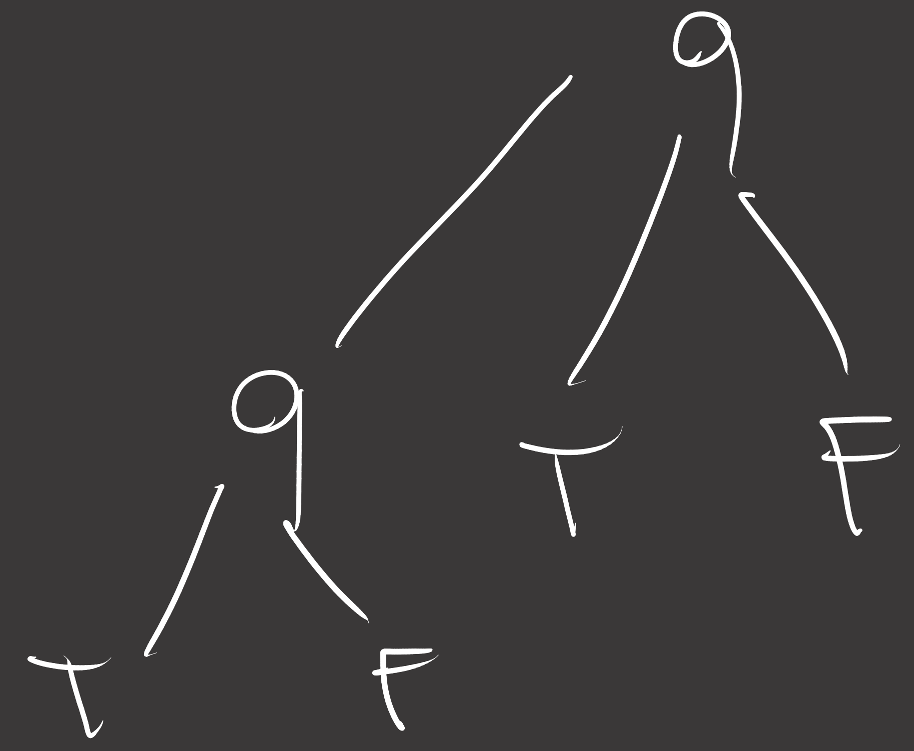
$A \times B$ is disj union



$A \Rightarrow B$



$\mathbb{B} \Rightarrow \mathbb{B}$



Plays

Def: A play on an arena $A = \langle M, \vdash, \lambda \rangle$ is a finite sequence of moves $p = (m_0, m_1, \dots, m_k)$ such that:

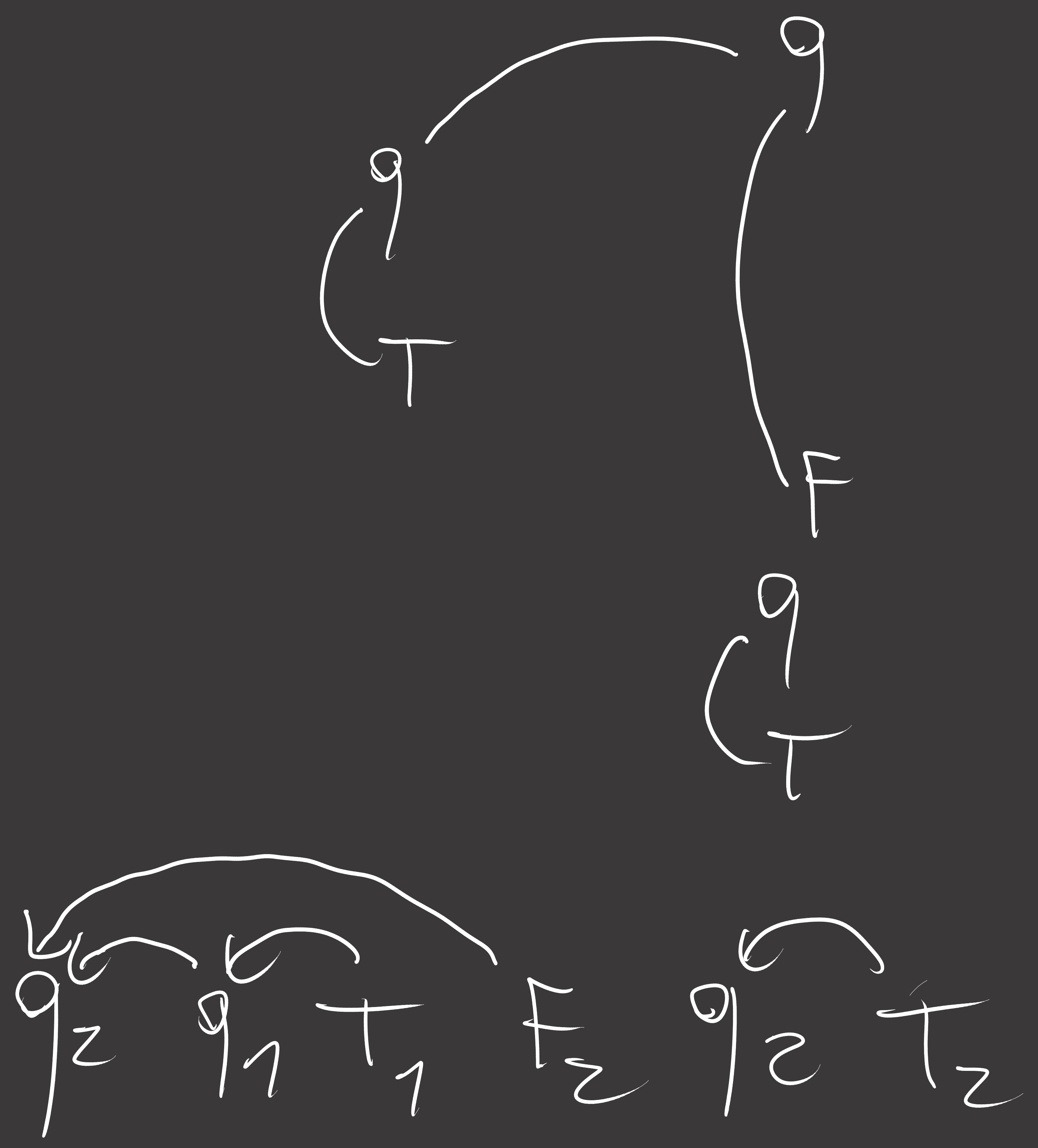
- $\lambda(m_0) = O$ and $\lambda(m_{i+1}) \neq \lambda(m_i)$
- $\forall i$, either $* \vdash m_i$
  or $\exists j < i$ such that $m_j \vdash m_i$

+ explicit pointers ⚠

Differences with game theory:

- there is no winner/loser, no payoff.
- it is possible to backtrack.
- it is not required to answer moves.

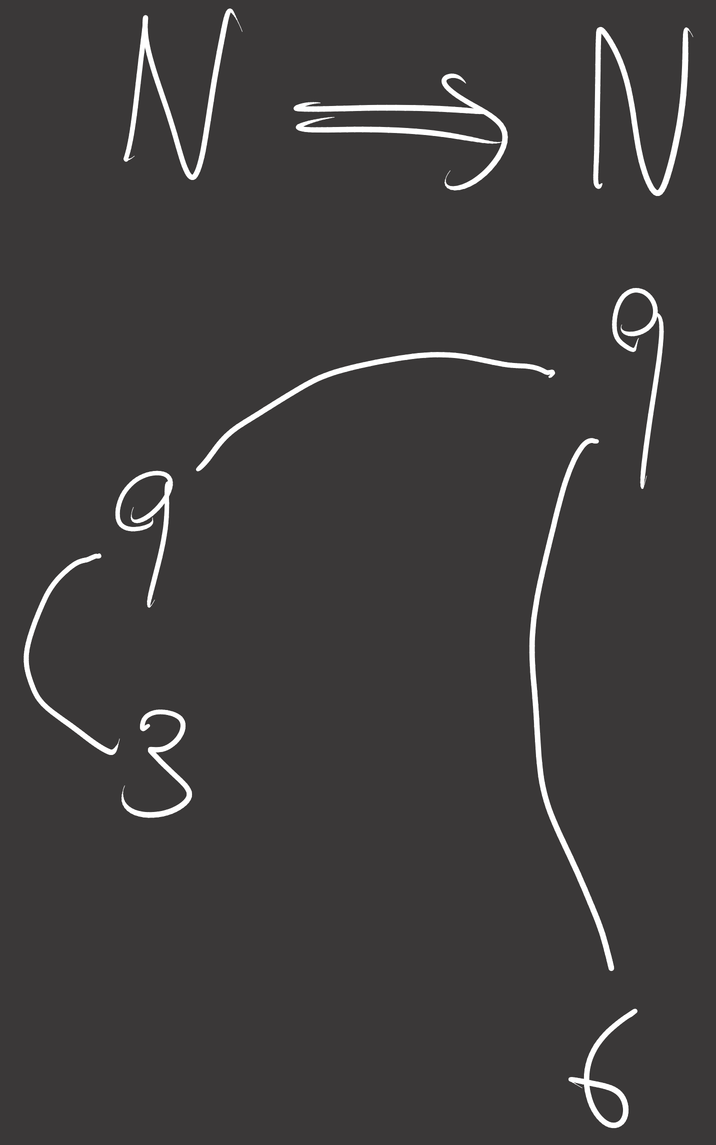
$$B \Rightarrow B$$



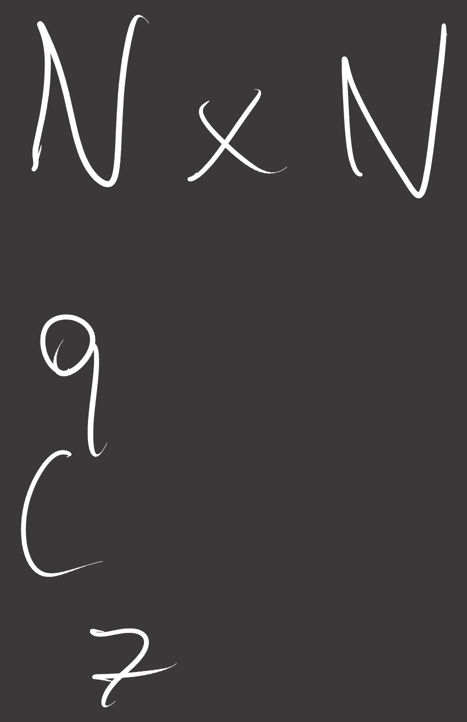
Examples of plays

- On N:

- On $N \Rightarrow N$:



- On $N \times N$:



P-Strategies (for player P)

Def: A strategy σ on an arena A is a non-empty set of plays, which is

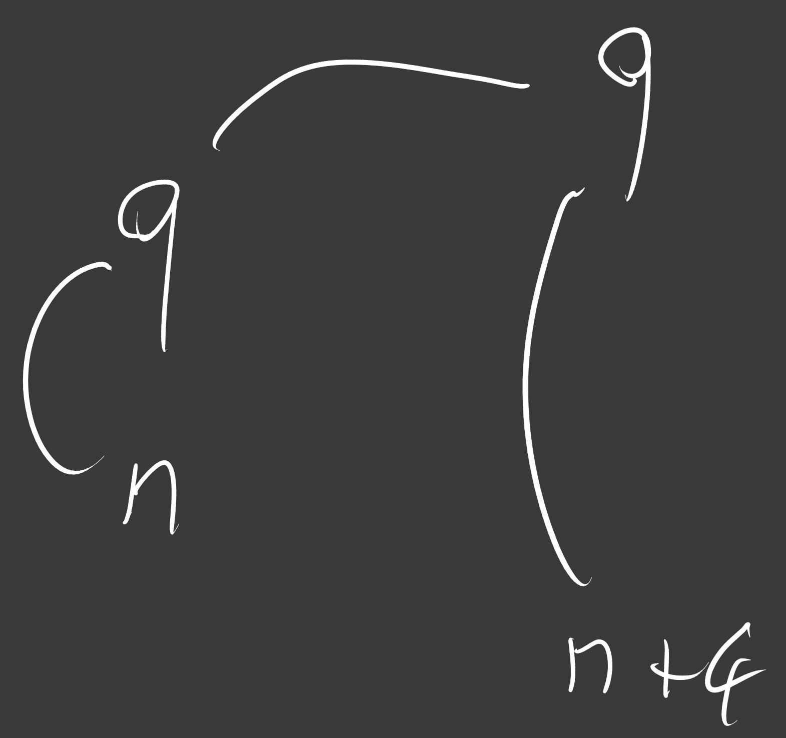
- closed under prefix
- receptive: if $p \in \sigma$ where $|p|$ is even, Opponent's turn
  then $pm \in \sigma$ for every possible m
- deterministic: if $pm_1 \in \sigma$ and $pm_2 \in \sigma$ where $|p|$ is odd, Player's turn
  then $m_1 = m_2$

ex: $\sigma = \{\varepsilon, q\}$

Examples on $N \Rightarrow N$

- $\lambda x. x+4$

$N \Rightarrow N$



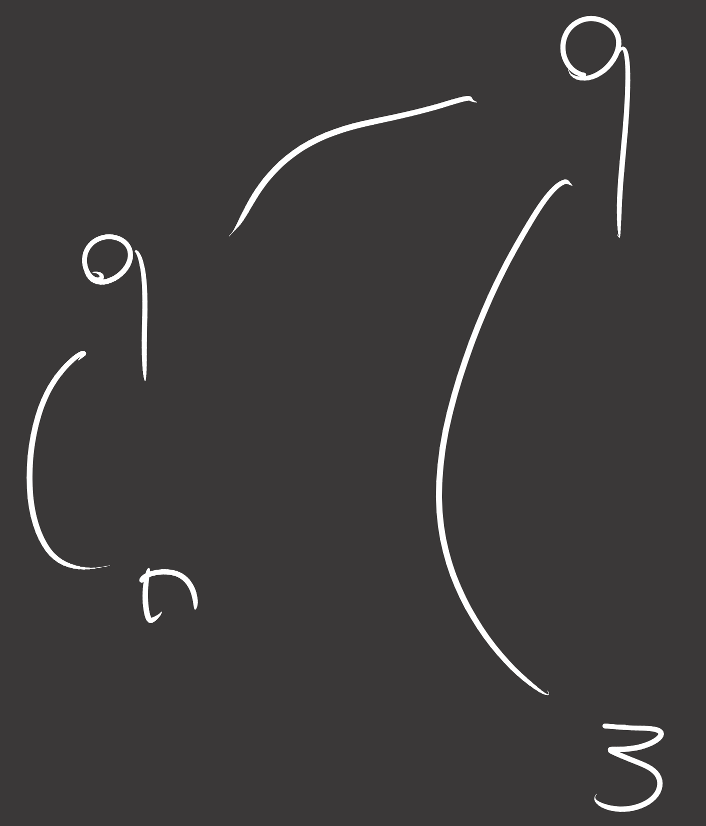
- $\lambda x. x * x$

$N \Rightarrow N$

- $\lambda x. 3$ vs $\lambda x.$ if $x=0$ then 3 else 3

$N \Rightarrow N$

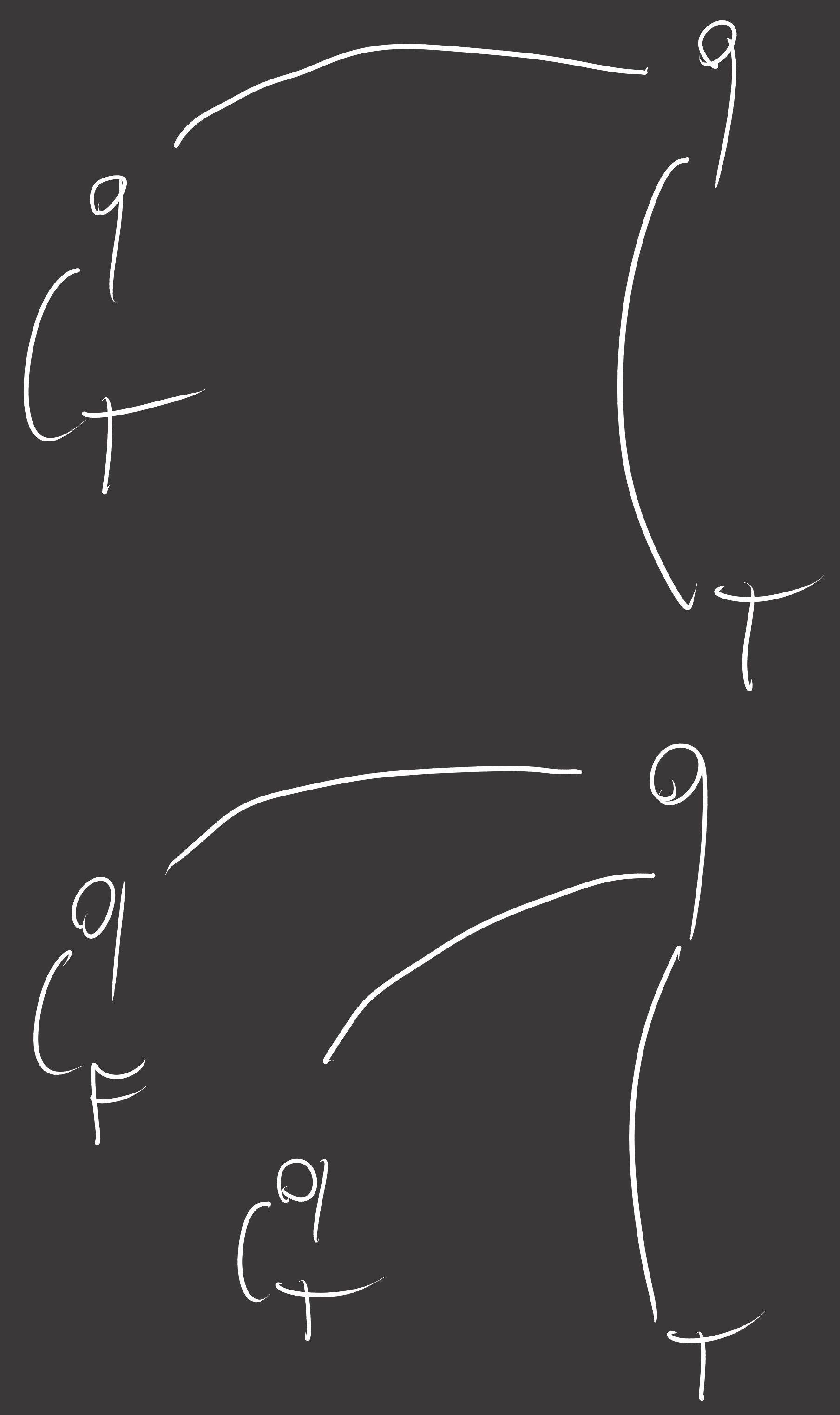
$N \Rightarrow N$



Examples on $B \times B \Rightarrow B$

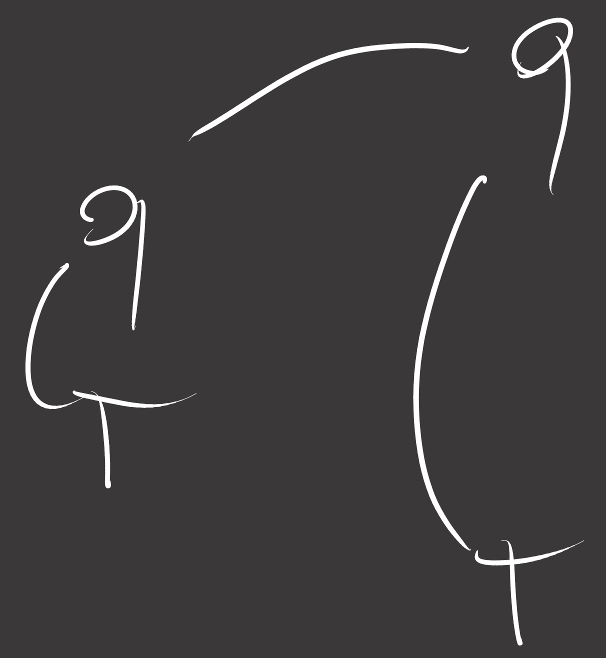
- left-or

$$B \times B \Rightarrow B$$



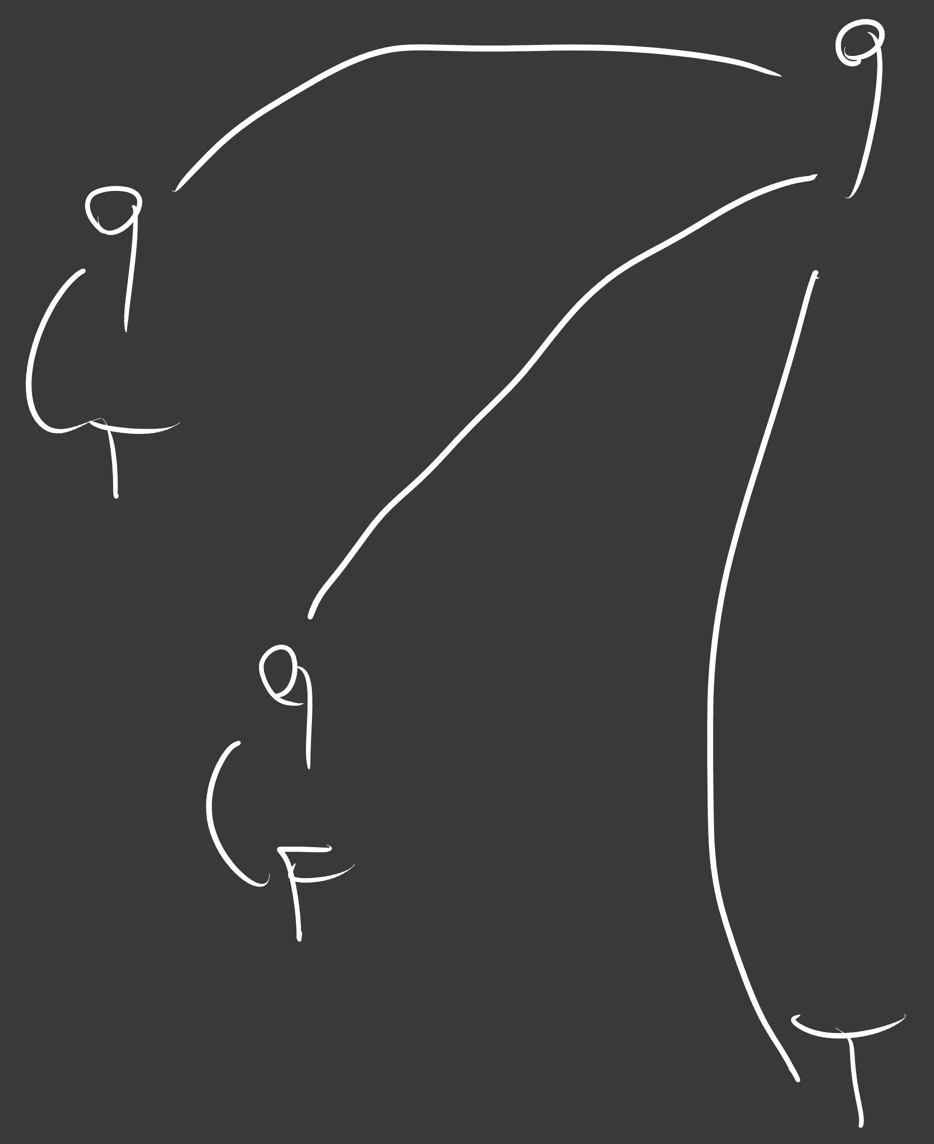
- right-or

$$B \times B \Rightarrow B$$



- left-right-or

$$B \times B \Rightarrow B$$

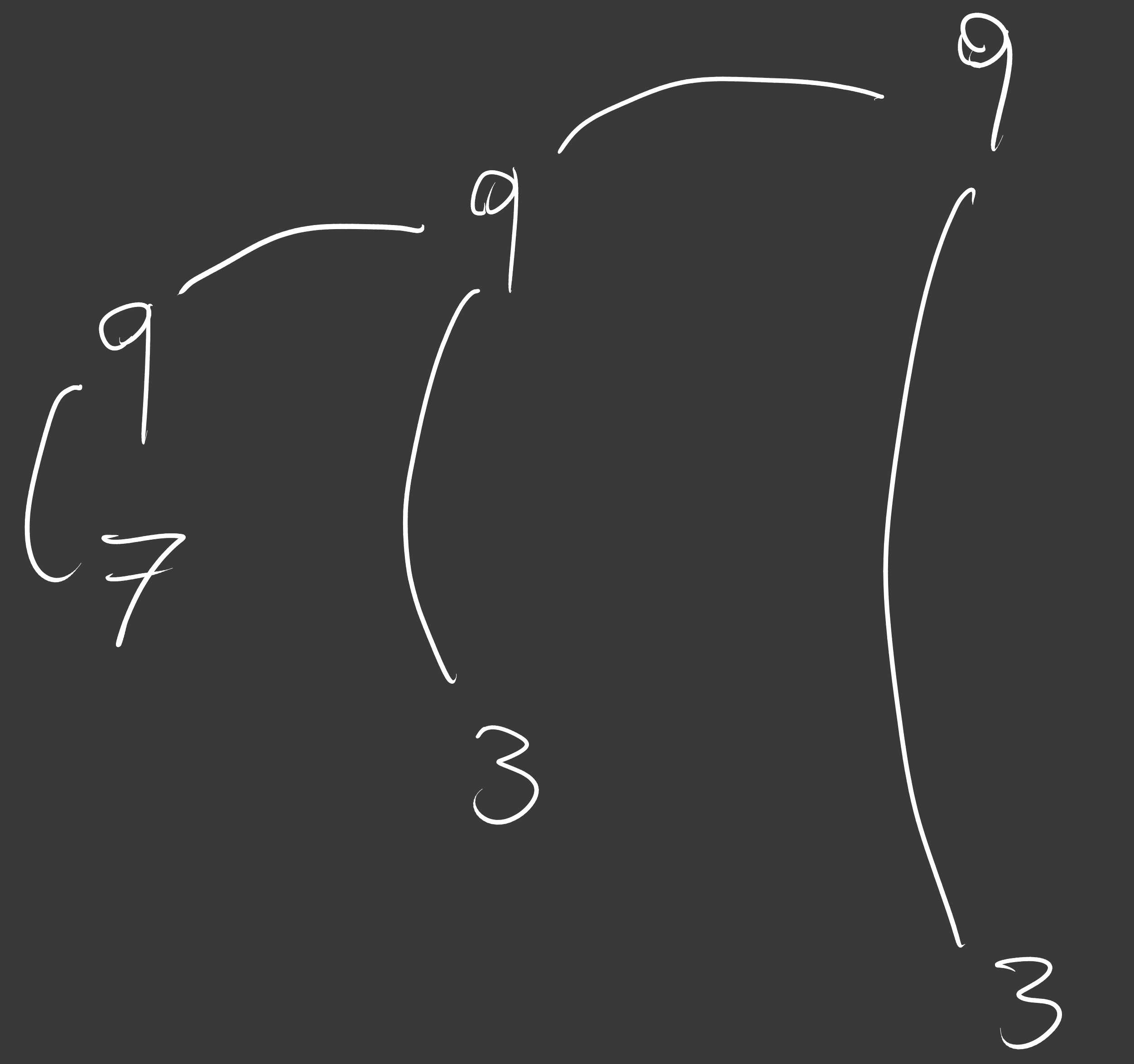


- right-left-or

Examples on $(N \Rightarrow N) \Rightarrow N$

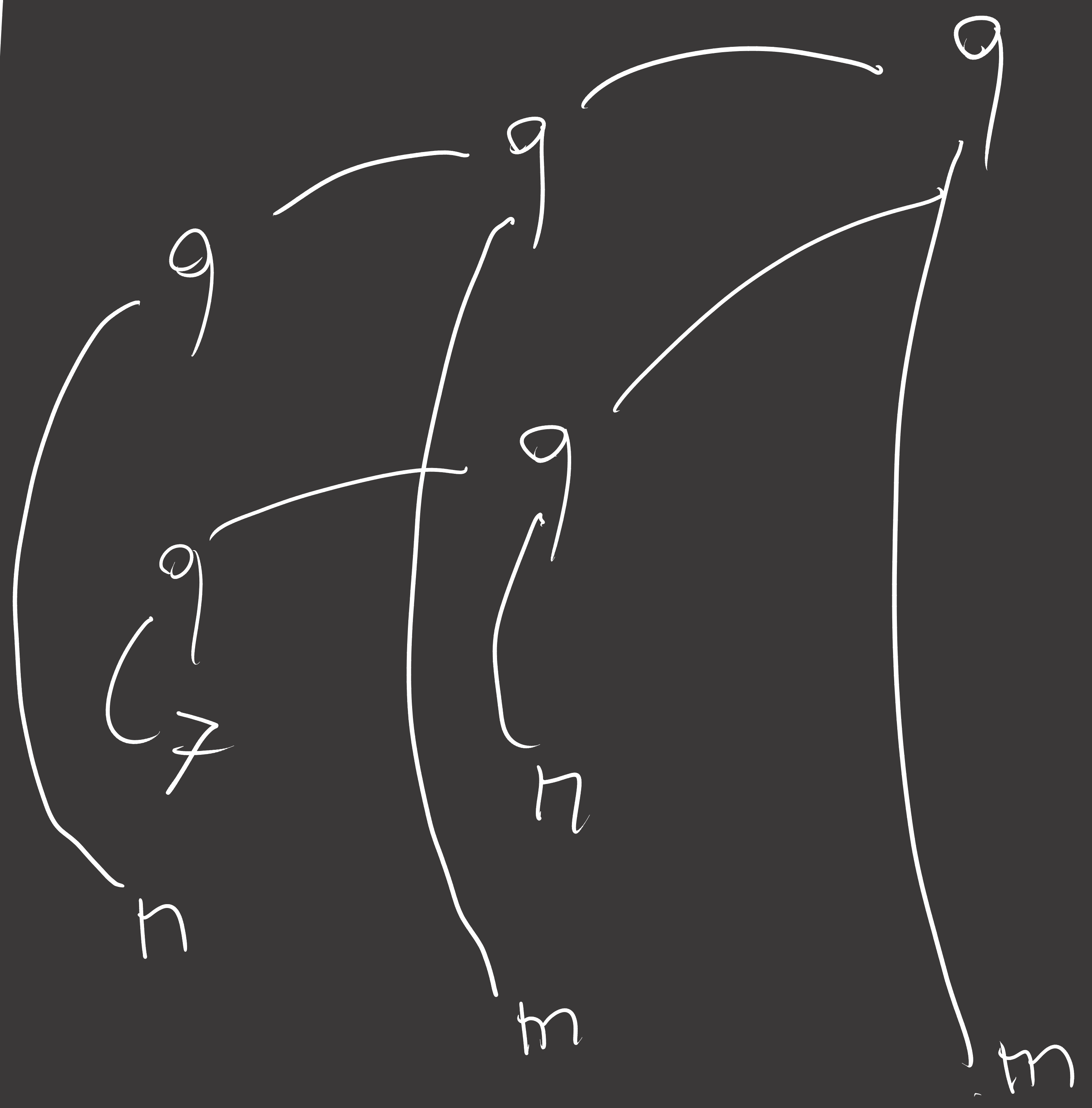
- $\lambda f.\ f\ 7$

$$(N \Rightarrow N) \Rightarrow N$$



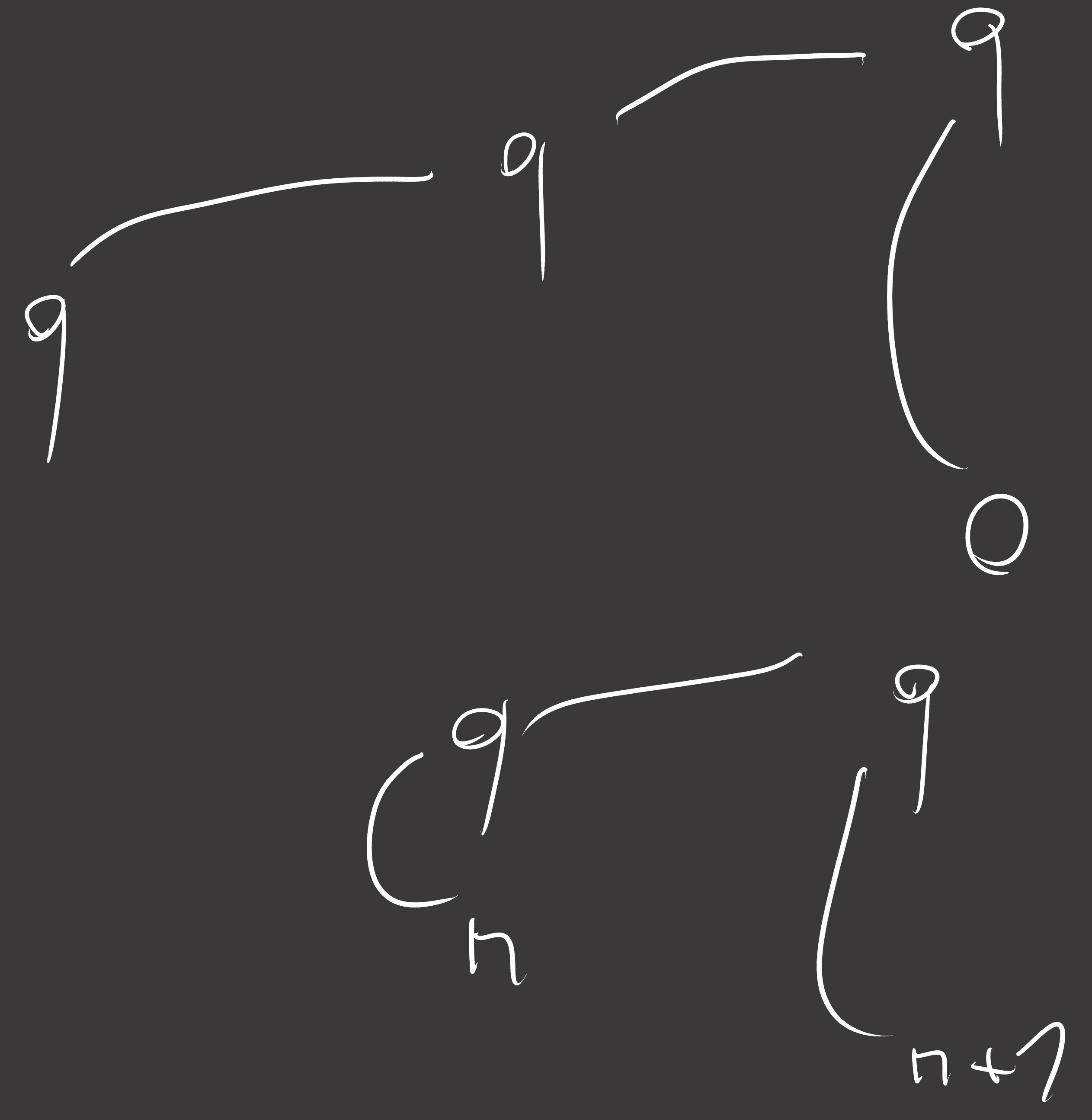
- $\lambda f.\ f\ (f\ 7)$

$$(N \Rightarrow N) \Rightarrow N$$



- catch

$$(N \Rightarrow N) \Rightarrow N$$



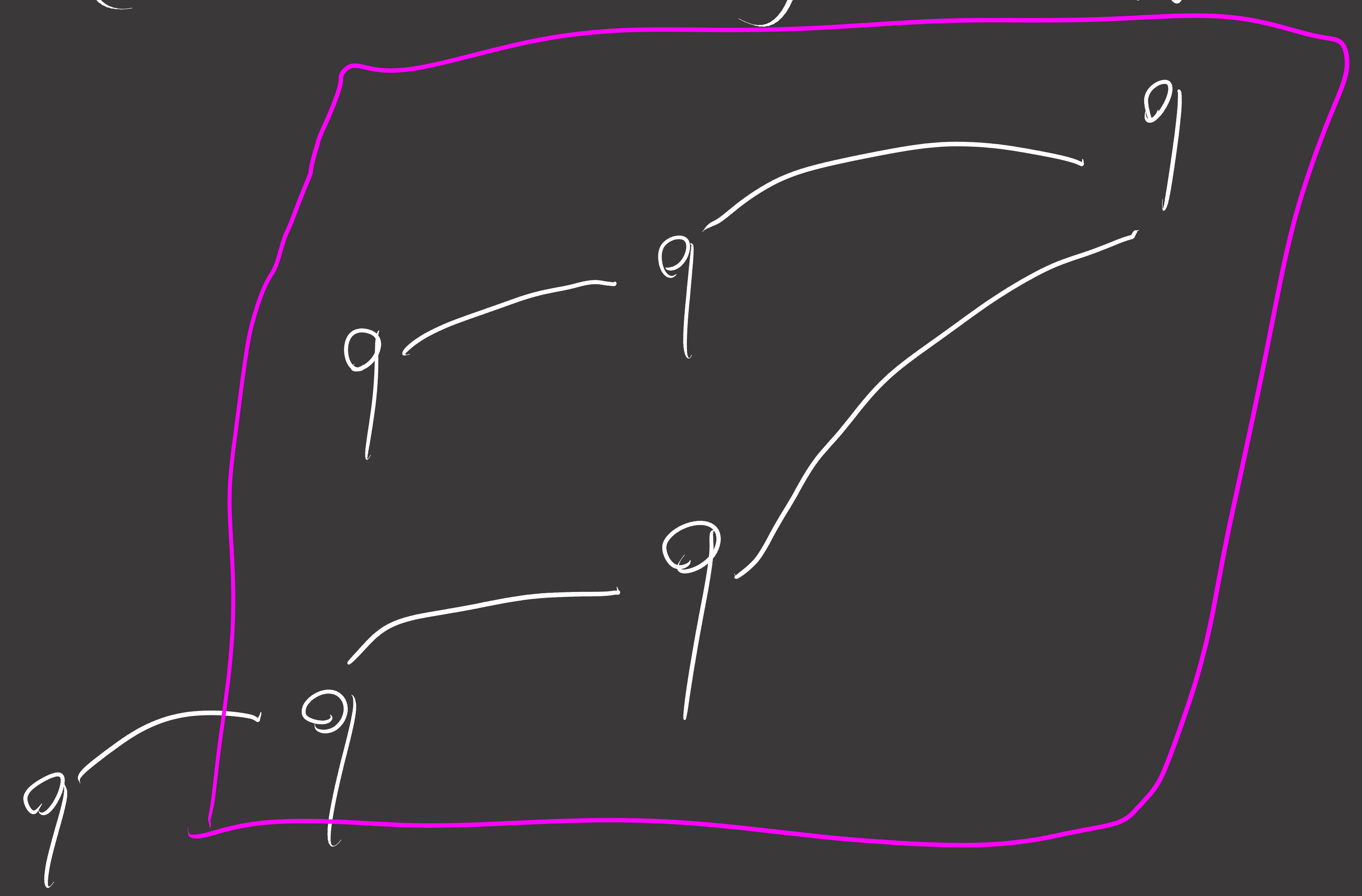
Example: the "or taster"

$$(B \times B \Rightarrow B) \Rightarrow B \qquad (B \times B \Rightarrow B) \Rightarrow B$$

Examples on $((N \Rightarrow N) \Rightarrow N) \Rightarrow N$

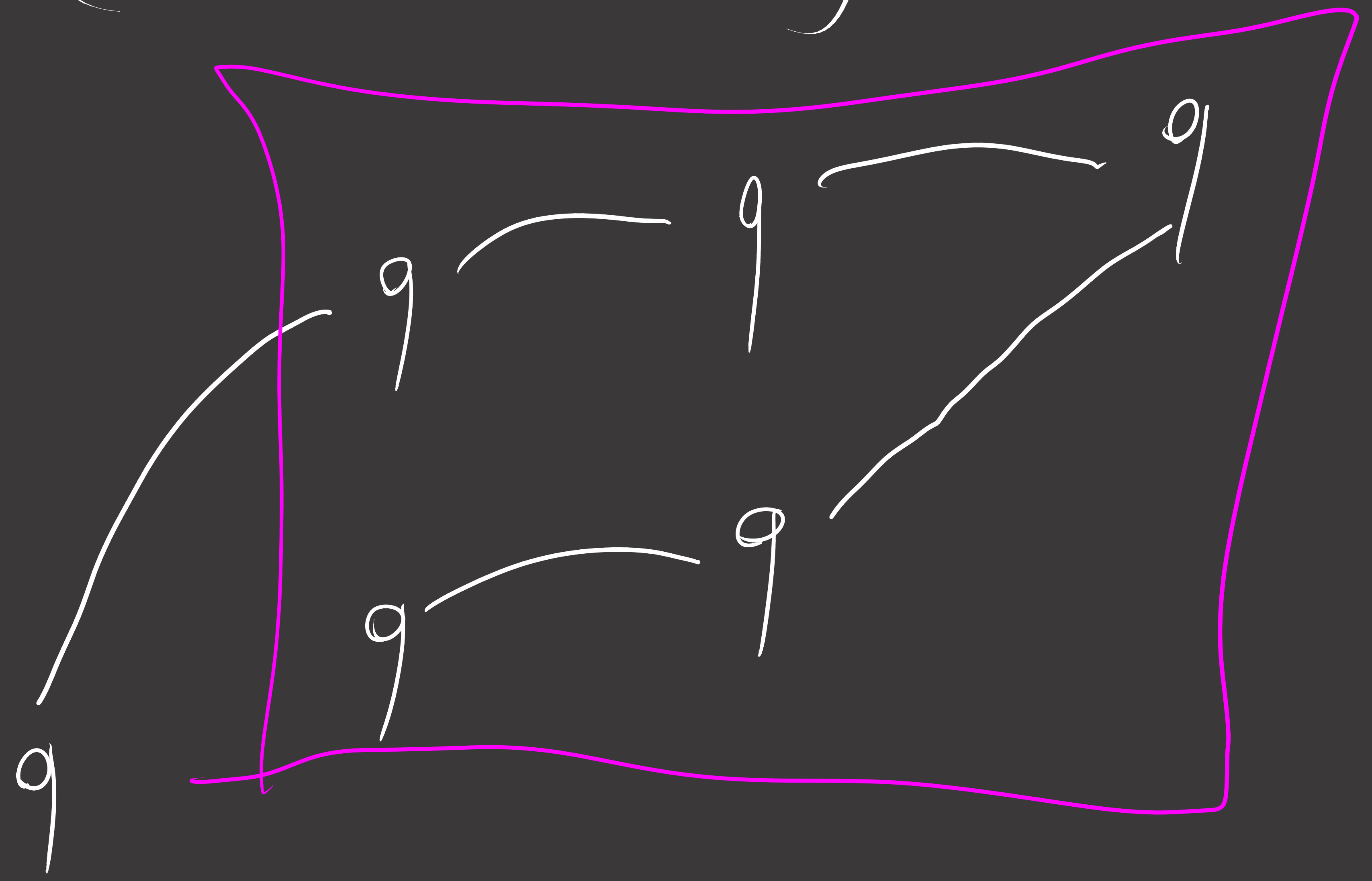
$$\lambda f.\, f\,(\lambda x.\, f\,(\lambda y.\, \boxed{y}))$$

$$((N \Rightarrow N) \Rightarrow N) \Rightarrow N$$



$$\lambda f.\, f\,(\lambda x.\, f\,(\lambda y.\, \boxed{x}))$$

$$((N \Rightarrow N) \Rightarrow N) \Rightarrow N$$

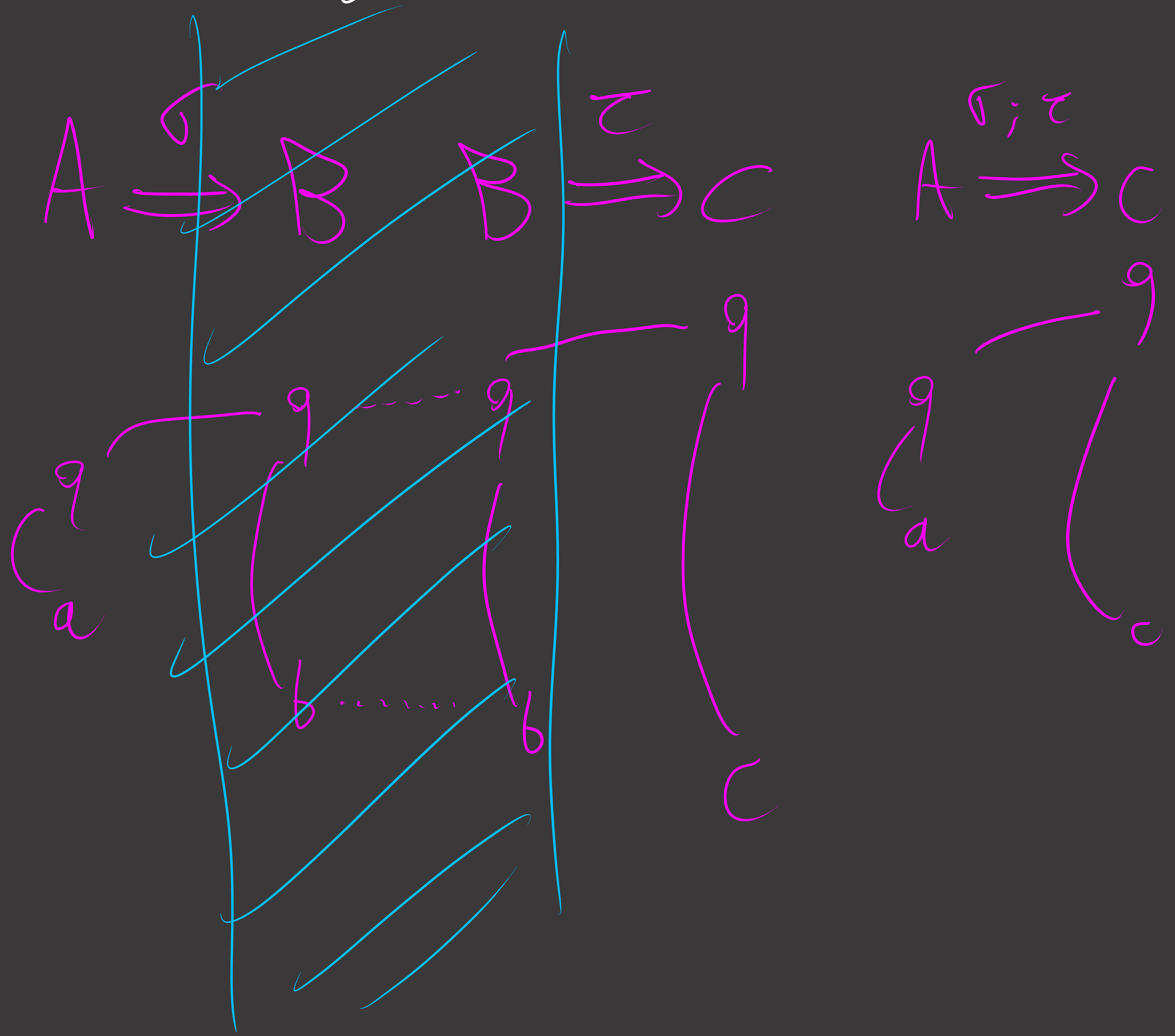


Category of arenas and strategies

Def: Let $\mathcal{G}$ be the category whose

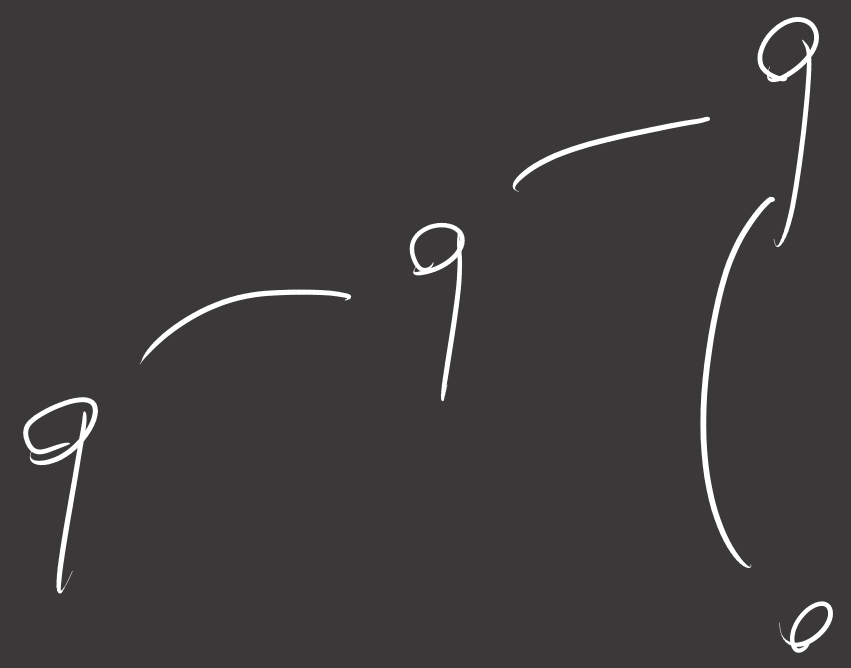
- objects are arenas
- morphisms $\sigma : A \to B$ are strategies on $A \Rightarrow B$

$id_A : A \Rightarrow A$



Properties of strategies

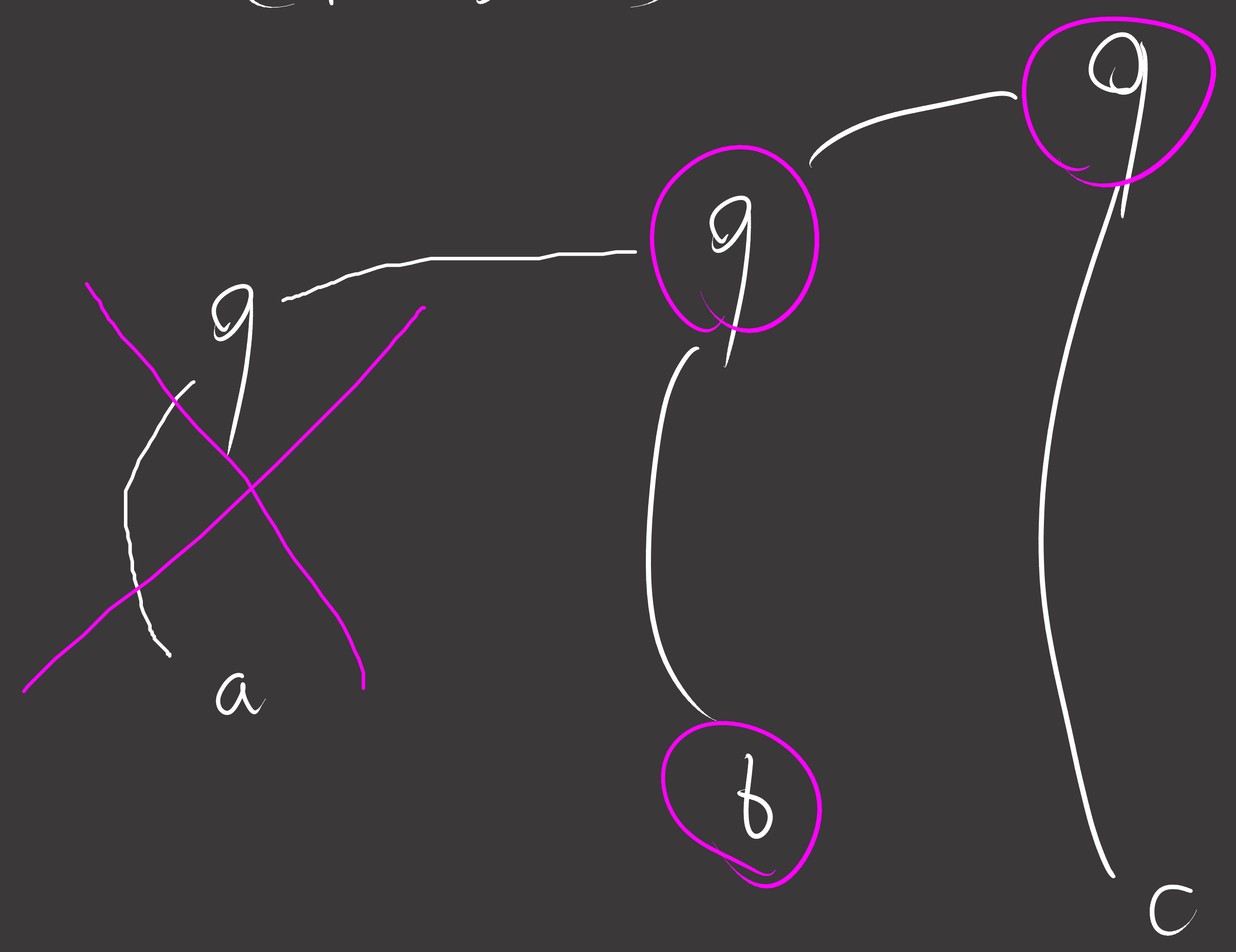
Def: A play p is well-bracketed if every answer points to the last pending question.
A strategy σ is well-bracketed if every $p \in \sigma$ is



Properties of strategies

Def: A play p is well-bracketed if every answer points to the last pending question.
A strategy σ is well-bracketed if every $p \in \sigma$ is

$$(A \Rightarrow B) \Rightarrow C$$



Def: Given a play p, the P-view $\ulcorner p \urcorner$ is defined by induction as follows:

$$\ulcorner \varepsilon \urcorner = \varepsilon$$

$$\ulcorner p\, m \urcorner = \ulcorner p \urcorner\, m \quad \text{if } m \text{ is a player move}$$

$$\ulcorner p\, m \urcorner = m \quad \text{if } * \vdash m$$

$$\ulcorner p_1\, \overset{\curvearrowleft}{m\, p_2\, n} \urcorner = \ulcorner p_1 \urcorner\, \overset{\curvearrowleft}{m\, n} \quad \text{if } n \text{ is an opponent move}$$

Def: A strategy σ is innocent if whenever $pm \in \sigma$

- m points to a move in $\ulcorner p \urcorner$
- if $p' \in \sigma$ with $\ulcorner p' \urcorner = \ulcorner p \urcorner$, then $p'm \in \sigma$

Fixpoints

$\mathcal{C}$ is enriched in $\mathbf{CPO}_\perp$

The game semantics for PCF

Let $\mathcal{G}_b$, $\mathcal{G}_i$ and $\mathcal{G}_{ib}$ be the sub-categories of $\mathcal{G}$ whose strategies are well-bracketed, innocent, and both.

Thm: They are cartesian-closed categories.

↳ for any PCF program $\Gamma \vdash t : A$,
we get a strategy $[\![t]\!] : [\![\Gamma]\!] \Rightarrow [\![A]\!]$ which is well-bracketed and innocent.

The game semantics for PCF

Let $\mathcal{G}_b$, $\mathcal{G}_i$ and $\mathcal{G}_{ib}$ be the sub-categories of $\mathcal{G}$ whose strategies are well-bracketed, innocent, and both.

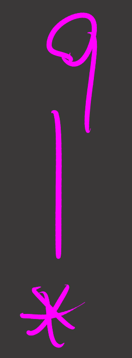
Thm: They are cartesian-closed categories.

↳ for any PCF program $\Gamma \vdash t : A$,
we get a strategy $[\![t]\!] : [\![\Gamma]\!] \Rightarrow [\![A]\!]$ which is well-bracketed and innocent.

Moreover, up to a "little" quotient:

Thm: The model $\mathcal{G}_{ib}/\approx$ is fully abstract : $t \equiv_{obs} u \iff [\![t]\!] \approx [\![u]\!]$

Factorization theorems

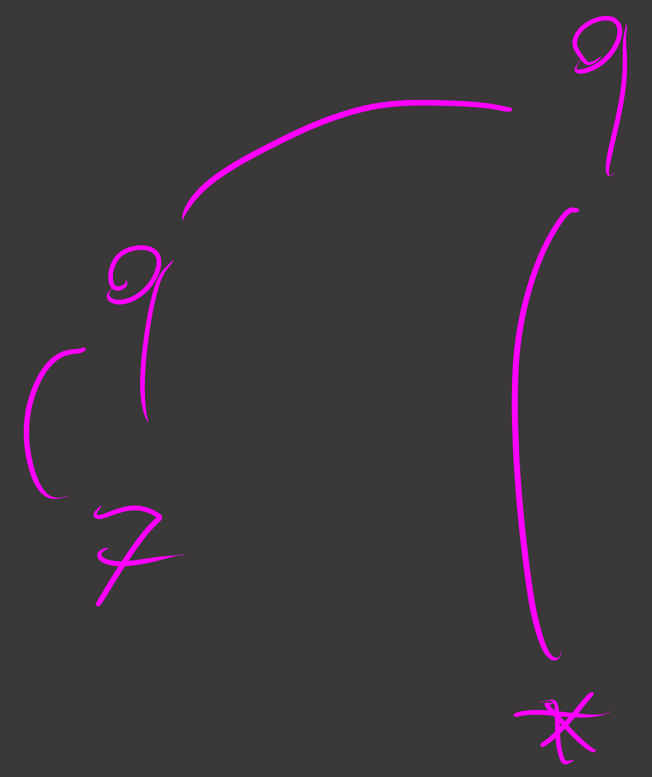
$U :$


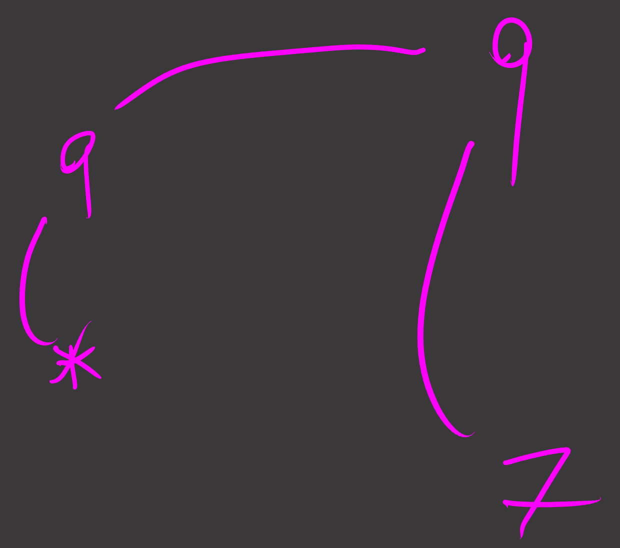
Let $\mathrm{Var} = (\mathbb{N} \overset{\text{write}}{\Rightarrow} U) \times (U \overset{\text{read}}{\Rightarrow} \mathbb{N})$

Thm: For every $\sigma : A$ in $\mathcal{G}_b$, there exists $\tau : \mathrm{Var} \Rightarrow A$ in $\mathcal{G}_{ib}$ such that $\sigma = \tau \circ \mathrm{cell}$.

Thm: $\mathcal{G}_b$ is a fully abstract model for PCF + references

$(\mathbb{N} \Rightarrow U) \times (U \Rightarrow \mathbb{N})$





Factorization theorems

Let $\mathrm{Var} = (N \Rightarrow U) \times (U \Rightarrow N)$

Thm: For every $\sigma : A$ in $\mathcal{G}_b$, there exists $\tau : \mathrm{Var} \Rightarrow A$ in $\mathcal{G}_{ib}$ such that $\sigma = \tau \circ \mathrm{cell}$.

Thm: $\mathcal{G}_b$ is a fully abstract model for PCF + references

Consider $\mathrm{catch}_k : (N_1 \times \dots \times N_k \Rightarrow N) \Rightarrow N$

Thm: For every $\sigma : A$ in $\mathcal{G}_i$, there exists $\tau : ((N_1 \times \dots \times N_k \Rightarrow N) \Rightarrow N) \Rightarrow A$ in $\mathcal{G}_{ib}$ such that $\sigma = \tau \circ \mathrm{catch}_k$

Thm: $\mathcal{G}_i/_{\approx}$ is a fully-abstract model of PCF + catch

—— adequate —— μPCF

Factorization theorems

Let $Var = (N \Rightarrow U) \times (U \Rightarrow N)$

Thm: For every $\sigma : A$ in $\mathcal{G}_b$, there exists $\tau : Var \Rightarrow A$ in $\mathcal{G}_{ib}$ such that $\sigma = \tau \circ cell$.

Thm: $\mathcal{G}_b$ is a fully abstract model for PCF + references

Consider $catch_k : (N_1 \times \dots \times N_k \Rightarrow N) \Rightarrow N$

Thm: For every $\sigma : A$ in $\mathcal{G}_i$, there exists $\tau : ((N_1 \times \dots \times N_k \Rightarrow N) \Rightarrow N) \Rightarrow A$ in $\mathcal{G}_{ib}$ such that $\sigma = \tau \circ catch_k$

Thm: $\mathcal{G}_i/_{\approx}$ is a fully-abstract model of PCF+catch

Questions?