Game Theory & Semi-Algebraic Geometry: A Symbiotic Relationship

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- In the past decade (especially), the opposite has been true as well: Sometimes, game theoretic objects display *universality* among semi-algebraic objects.
- There are relevant (not yet fully pursued) links to complexity of Nash equilibrium computation.


Nash
Equilibria
Semi-algebraic
Sets
"Anything you can do, I can do better...."

Strategic Form Games

A game (in strategic form) consist of:

- A finite set I of players
- Finite action spaces $A^1, \ldots, A^I$
- A mapping $G : \prod_{i \in I} A^i \to \mathbb{R}^I$ assigning, for each action profile, a payoff to each player.

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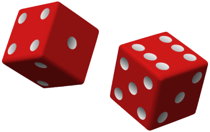
Player i's mixed strategies are $\Delta(A^i)$, the distributions over A^i. Under a profile of mixed action $z^1, \ldots, z^I$, the payoff is:



$$G(z^1, \ldots, z^I) = E_{a^k \sim z^k} G[a^1, \ldots, a^I] = \sum_{a=(a^1,\ldots,a^I) \in \prod_{k=1}^I A^I} G(a) \cdot \prod_{k=1}^{I} z^k[a^k]$$

Nash Equilibria

A Nash equilibrium is a profile of mixed actions $z^1, \ldots, z^I$ such that no player has an incentive to deviate. I.e.,

$$G^j(z) \geq G^j(b, z^{-j}), \ \forall j \in I, b \in A^j$$

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Trivial observation: The set of Nash equilibrium is a compact subset of $\prod_{i \in I} \mathbb{R}^{A^i}$ defined by polynomial equalities / inequalities.

$$z^j[b] \geq 0, \ \forall j \in I, b \in A^j$$

$$\sum_{b \in A^j} z^j[b] = 1, \ \forall j \in I$$

$$\sum_{a \in \prod (A^i)_{i \in I}} (\prod_{i \in I} z^i[a^i]) G^j(a) \geq \sum_{a \in \prod (A^i)_{i \in I}, a^j = b} \Big(\prod_{i \in I, i \neq j} z^i[a^i] \Big) G^j(a), \ \forall j \in I, b \in A^j$$

Existence of Nash Equilibria

Theorem (Nash (1951))

Every (finite, strategic form) game possesses an equilibrium.

Proof: Brouwer's theorem.

Examples

- Prisoner's dilemma:

$3, 3$	$0, 4$
$4, 0$	$1, 1$

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$1,-1$	$-1,\ 1$
$-1,\ 1$	$1,-1$

The unique equilibrium is $(\frac{1}{2},\frac{1}{2}) \times (\frac{1}{2},\frac{1}{2})$.

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  The unique equilibrium is $(\frac{1}{2}, \frac{1}{2}) \times (\frac{1}{2}, \frac{1}{2})$.

- Coordination game:

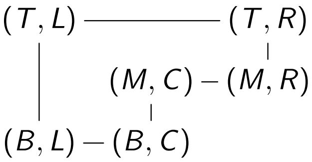
$2,2$	$0,0$
$0,0$	$1,1$

  The equilibria are (Top, Left), (Bottom, Right), and $(\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})$.

Example: Kohlberg-Mertens (1986)

Set of equilibria is S^1.

	L	C	R
T	$1, 1$	$0, -1$	$-1, 1$
M	$-1, 0$	$0, 0$	$-1, 0$
B	$1, -1$	$0, -1$	$-2, -2$



Sets of Nash Equilibria: Topologically, Anything Goes

Theorem ('Folk Conjecture'; Balkenborg, Vermeulen (2019))

Every compact simplicial complex is homeomorphic to the set of Nash equilibria of some game.

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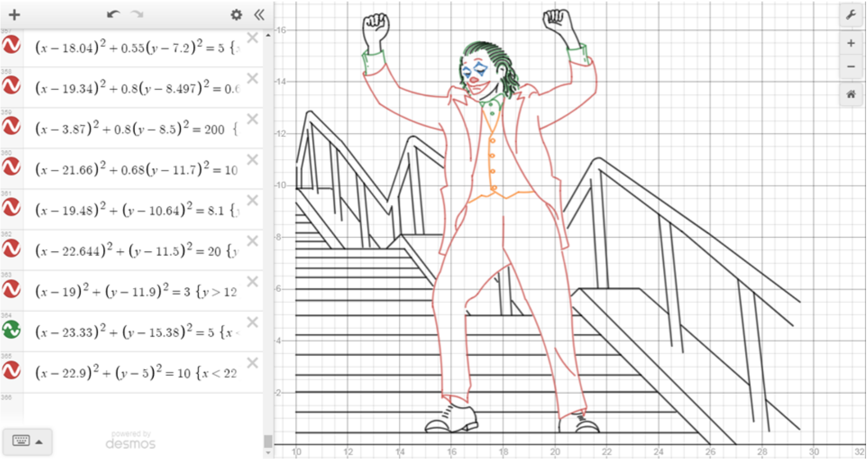
Open question: What about diffeomorphic?

Semi-Algebraic Sets

A set $X \subseteq \mathbb{R}^N$ is *semi-algebraic* it can be defined via a boolean combination of polynomial equalities and inequalities,

$$\{(x,y) \in \mathbf{R}^2 \mid x^2/25 + y^2/16 < 1 \text{ and } x^2 + 4x + y^2 - 2y > -4 \text{ and } x^2 - 4x + y^2 - 2y > -4 \text{ and } (x^2 + y^2 - 2y \neq 8 \text{ or } y > -1)\}$$

(Source:Bochnak et al.)



(Source: Reddit)

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- The collection of semi-algebraic sets is closed under finite unions, finite intersects, and complements.

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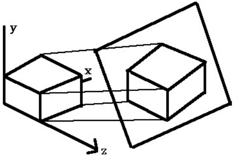

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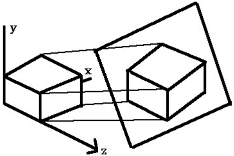

- The set of Nash equilibria (of mixed extension of finite game $G : \prod_{i \in I} A^i \to \mathbb{R}^I$) is always non-empty compact and semi-algebraic:

$$z^j[b] \geq 0, \ \forall j \in I, b \in A^j$$

$$\sum_{b \in A^j} z^j[b] = 1, \ \forall j \in I$$

$$\sum_{a \in \prod (A^i)_{i \in I}} (\prod_{i \in I} z^i[a^i]) G^j(a) \geq \sum_{a \in \prod (A^i)_{i \in I}, a^j = b} \Big(\prod_{i \in I, i \neq j} z^i[a^i] \Big) G^j(a), \ \forall j \in I, b \in A^j$$

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- Hence, so do Nash equilibria.
- In fact, can bound components by number of players and strategies, due to bounds (Thom, Milnor; also Warren '68).

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- Projections correspond to ∃-quantifer (over reals); since s.a. sets closed to complements, we have ∀-quantier as well.

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- Example: Closure of a semi-algebraic set is s.a.. If formula ϕ built from polynomial in/equalities defines $A \subseteq \mathbb{R}^N$, then $\overline{A}$ defined by

$$\overline{\phi}(x) := \forall \varepsilon > 0,\ \exists y \in \mathbb{R}^N, (\phi(y) \wedge \sum (y_i - x_i)^2 < \varepsilon^2)$$

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- Example: The supremum/infimum of a parametrized s.a. function $u(\lambda, \cdot) : X \to \mathbb{R}$ is s.a. with X s.a., if ϕ defines $Graph(u)$,

$$(v = \sup_{x \in X} u(\lambda, x)) \Leftrightarrow (\forall y \in X, \exists t \in \mathbb{R},\ \phi(\lambda, y, t) \wedge t \leq v)$$
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- Hence, the value (minmax=maxmin) of $u(\lambda, \cdot, \cdot)$ is a s.a. function of $\lambda \in \mathbb{R}$.

More Niceness

- S.a. sets have well-defined dimension; no 'weird' sets; finite union of 'nicely' embedded cubes/simplices.
- Single-variable functions display piece-wise monotonicity.

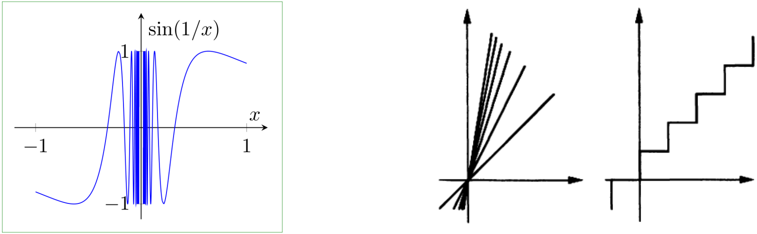

Figure: Not Semi-Algebraic. (Source (right): Bochnak et al.)

- In particular, complement of 'generic' set is of lower dimension.
- S.a. functions differentiable except for degenerate set.

Zero-Sum Stochastic Games (Bewley-Kohlberg, '76)

- In a stochastic games (Shapley, '53), the state $\in S$ evolves stochastically as a function of state and actions.
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- Assume S, actions finite.
- The condition of v_λ being the value is semi-algebraic in λ,

$$v_\lambda(s) = val\Big(\lambda r(s,\cdot,\cdot) + (1-\lambda)\sum_{s'\in S} q(s' \mid s,\cdot,\cdot)v_\lambda(s')\Big)$$

- Hence, $v_\lambda(s)$ is monotonic in nbd of $0 \to \lim v_\lambda$ exists.
- **Open Question:** What if action spaces compact s.a. / data s.a.?
- **Open Question:** Which functions can v_λ be? (Lehrer et al 2016 analyze the single-player case, MDP).

Generic Finiteness Govindan & Wilson (2001)

- Harsanyi (1973), Rosenmuller (1971), Wilson (1971) and others show equilibria are generically finite.
- Nice proof, based on fact: $f : X \to Y$ continuous s.a., $dim(X) = dim(Y)$, then $f^{-1}(y)$ generically finite.

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- Nice proof, based on fact: $f : X \to Y$ continuous s.a., $dim(X) = dim(Y)$, then $f^{-1}(y)$ generically finite.
- Enough to show finiteness of full support eq:
  - Define polynomial map p on space $Games \times Strategy\ Profiles \times Payoffs$ s.t. Dp is non-singular on $p^{-1}(0) =$ is completely mixed Nash eq manifold, s.a. of dimension $= dim(Games)$.
  - Applying fact to projection $p^{-1}(0) \to Games$ gives result.

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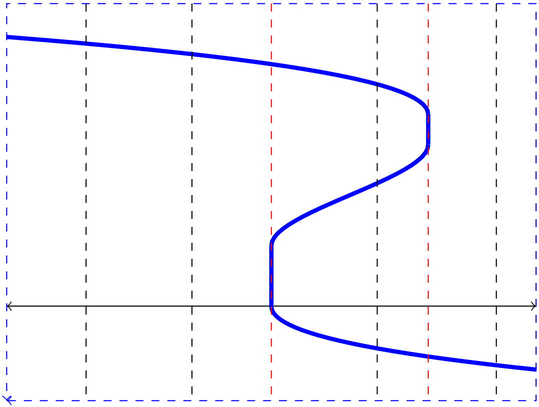

Figure: The Equilibrium Manifold $p^{-1}(0)$ of dimension $= dim(Games)$
$p^{-1}(0) \subseteq$ Games $\times$ Strategy Profiles $\times$ Payoffs

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- What if we know payoffs are semi-algebraic, and how much information is needed to define their graphs?
- Allows us to bound pseudo-dimension of payoff functions, to deduce....

Approximate Equilibria, L (2021)

Theorem

There is a function $\phi(\varepsilon, N, m, r, s)$,

$$\phi = \frac{512}{\varepsilon^2}\Big(\ln(32N) - \ln(\varepsilon) + 2\cdot(m+1)\cdot\log_2(8er(s+1))\cdot\ln\big(\frac{64e}{\varepsilon}\ln(\frac{64e}{\varepsilon})\big)\Big)$$

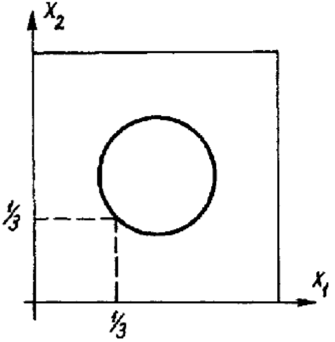
s.t. every game with N *players, actions in* $\mathbb{R}^m$*, payoffs* $\in [0,1]$*, s.a. payoff function graphs defined by* s *polynomials of degrees* $\leq r$ *possess* ε*-equilibria w/ support size* $\leq \phi(\varepsilon, N, m, r, s)$.

In fact, denoting $k = \phi(\varepsilon, N, m, r, s)$, the strategies are k-uniform: Weights are multiples of $\frac{1}{k}$. Via random sampling.

Complexity of Solutions

- In bimatrix games with rational payoffs, there is at least one equilibrium with rational components, hence rational payoff.
- Not so for 3 player games (Nash).
- In fact (Bubelis, '79) for *any* algebraic number α, there is a 3-player game with rational payoffs and unique eq, which has payoff α for some player.
- Bubelis also hints at complexity of equilibria set, 6-player game with circular equilibria (Kohlberg-Mertens '86 do it with 2 players up-to-homeomorphism).

Circular Example (Bubelis, '79)



$$(1-3x_1)\left(\left(x_2-\frac{1}{2}\right)^2+\left(x_1-\frac{1}{2}\right)^2-\frac{1}{18}\right)=0$$

$$(1-3x_2)\left(\left(x_2-\frac{1}{2}\right)^2+\left(x_1-\frac{1}{2}\right)^2-\frac{1}{18}\right)=0$$

$$x_3=\frac{1}{3}(x_1+1) \qquad x_4=\frac{1}{10}(x_2^2-x_2+3)$$

$$x_5=\frac{1}{10}(x_1^2-x_1+3) \qquad x_6=\frac{1}{3}(x_2+1)$$

Bimatrix Games & Many Players

- For bimatrix games, the set of Nash eq is a finite disjoint union of sets $P_1 \times P_2$, where P_1, P_2 are polytopes, $P_1 \times P_2$ *exchangeable*: $(p, q), (p', q') \in P_1 \times P_2 \to (p', q), (p, q') \in P_1 \times P_2$ (e.g., Voroben, '58, Kuhn, '61)

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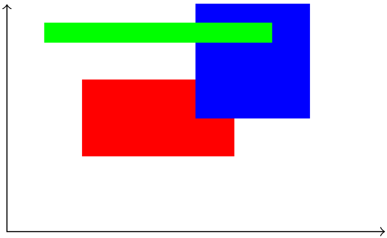
- $\rightarrow$ The set of Nash eq payoffs $=$ union of rectangles (parallel to axis).



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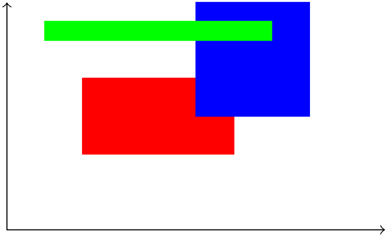

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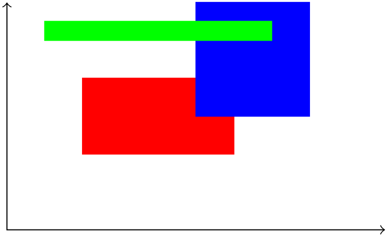

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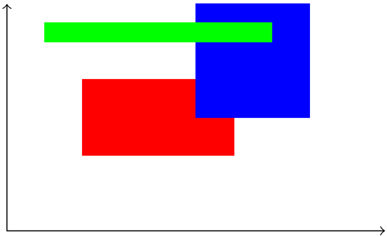

- (Lehrer et al, 2011) Converse: Any finite union of such rectangles = Nash eq of some bimatrix game.
- What if ≥ 3 players?No way!

Theorem (Vigeral (in preperation))

For any $N \geq 3$ and any compact non-empty semi-algebraic subset $C \subseteq \mathbb{R}^N$, there is an N-player game G s.t. the set of Nash equilibrium payoffs of G is C, i.e., $NEP(G) = C$.

Thm 1 of L (2016); also Vigeral & Viossat (2016)

Let I be a finite set of players, with finite action sets $A^1, \ldots, A^I$.

Theorem

Let $\emptyset \neq X \subseteq \prod_{i \in I} \Delta(A^i) \subseteq \prod_{i \in I} \mathbb{R}^{A^i}$ *be compact and semi-algebraic. Then there exists a collection* $\mathcal{P}$ *of binary players (i.e., players w/ action set* $\{0,1\}$*) and a game* G *on the set of players* $I \cup \mathcal{P}$*, such that* X *is the projection of the set of equilibria of* G*; i.e.,*

$$X = \{(z^i)_{i \in I} \mid z \in \prod_{i \in I} \Delta(A^i) \times \prod_{j \in \mathcal{P}} \Delta(\{0,1\}) \text{ is an equilibrium of } G\}$$

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i.e., if $pr : \mathbb{R}^{\sum A^i + 2\mathcal{P}} \to \mathbb{R}^{\sum A^i}$ is the projection,

$$X = pr(\mathcal{NE}(G))$$

(Strengthening of) Thm 2 of L (2016)

Theorem

Let $A \subseteq \mathbb{R}^M$ *be semi-algebraic; and let* $g : A \to [0,1]^K$ *be a continuous semi-algebraic function. Then there exists an affine embedding* T *from* $\mathbb{R}^M$ *to the space of games with some* $K + J$ *binary players s.t.*

$$pr_K(\mathcal{N}\mathcal{E}(T(\cdot))) = g(\cdot)$$

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i.e., for each $p \in A$, and all $z \in \mathcal{NE}(T(p))$, $z^i = g^i(p)$ for $i \in K$.

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Later work (in preperation) applies this to understanding fixed point correspondences, and develops a *universality* result as well....

Examples: The Identity, Addition

- Take $A = (0,1)$, observe $id : (0,1) \to (0,1)$. Observe

$$H(q) = \begin{array}{|c|c|}\hline 1,-1 & 1-4q, 3-4q \\ \hline 4q-3, 4q-1 & 1,-1 \\ \hline \end{array}$$

For $0 < q < 1$, the unique equilibrium is $(q, 1-q) \times (q, 1-q)$.

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$$H(q) = \begin{array}{|c|c|} \hline 1, -1 & 1-4q, 3-4q \\ \hline 4q-3, 4q-1 & 1, -1 \\ \hline \end{array}$$

  For $0 < q < 1$, the unique equilibrium is $(q, 1-q) \times (q, 1-q)$.

- Let $B = (0, \frac{1}{2}) \times (0, \frac{1}{2})$, observe the addition function $(x, y) \to x + y$ on $B \to (0, 1)$. Let α, β play $H(x + y)$.

Examples: $x^2 + y^2$

Let $f(x, y) = x^2 + y^2$ on $(0, \frac{\sqrt{2}}{2}) \times (0, \frac{\sqrt{2}}{2}) \to (0, 1)$. Define 6 players:

- α^1, β^1 play $H(x)$; in eq., both play $(x, 1-x)$
- α^2, β^2 play $H(y)$; in eq., both play $(y, 1-y)$.
- α, β play $H(u^{\alpha_1} \cdot u^{\beta_1} + u^{\alpha_2} \cdot u^{\beta_2})$, where u^p denotes the mixed action of player p.

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- One can continue this way to represent any polynomial (in any bounded set), and show that the set of representable functions is closed under composition....
- ...then use a trick for all semi-algebraic functions.

Relation to Complexity of Nash Eq

- Very similar 'gadgets' constructed for simple operations by Daskalakis Goldberg Papadimitriou (2009), *The Complexity of Computing a Nash Equilibrium*.

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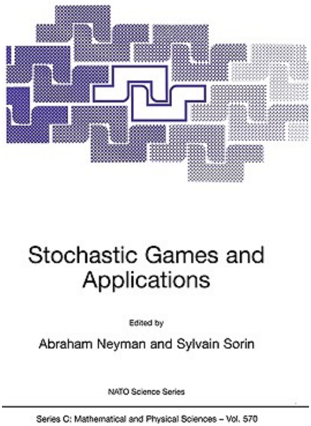
- Construction there for less general functions, but accounts for ε-equilibria.
- Analysis of L (2016) construction probably yields similar complexity results.

Super Short Summary

- Semi-algebraic sets and functions are 'nice'.
- Their properties have been, and continue to be, used in game-theoretical applications.
- More recently, it's been shown that Nash equilibria and the Nash equilibrium correspondences show certain *universality* properties among s.a. sets, functions, and more.
- There are relations to complexity of structure/computing Nash equilibria to be explored.

To Learn More...



See: Chapter 6, *"Real algebraic tools in stochastic games"*

Thank You!

Questions?