A Dual Adjunction Between Ω-Automata and Wilke Algebras

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MSP 101

Automata theory

Automata are abstract models of computation.

Formal languages are sets of words that can describe valid syntax, successful computation, etc.

Some types of automata correspond to classes of formal languages.

Types of automata	Classes of formal languages
Finite-state automata	Regular languages
Pushdown automata	Context-free languages
Turing machines	Decidable languages

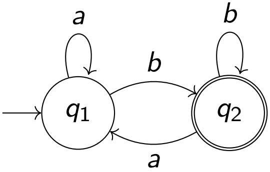
Automata for infinite words

Can we characterise languages of infinite words with automata?

Definition (Büchi, 1962)

A Büchi automaton is a tuple $A = (\Sigma, Q, S, \Delta, F)$, where:

- Σ is a finite alphabet;
- Q is a finite set of states;
- $S \subseteq Q$, a set of initial states;
- $\Delta \subseteq Q \times \Sigma \times Q$, a transition relation;
- $F \subseteq Q$, a set of final states.



An infinite word α is accepted by A, if there exists a run of α that starts in an initial state and visits some final state infinitely often.

Regular languages of infinite words

Languages accepted by Büchi automata are called ω-regular.

Definition

Given an ω-regular language L, we write $UP(L)$ for the set of its ultimately periodic words, i.e., words of the form $uv^\omega := uvvv\ldots$.

Proposition

If L_1 and L_2 are ω-regular languages, then $UP(L_1) = UP(L_2)$ implies $L_1 = L_2$.

Lasso languages

(Ciancia, Venema, 2012, ideas from Calbrix et al., 1993)

A lasso is a pair of finite words (u, v) with $|v| > 0$.
The lasso (u, v) represents the infinite word uv^ω.

A lasso language is a set of lassos.
If L is ω-regular, then L is uniquely determined by the lasso language:

$$Lasso(L) := \{(u, v) \mid uv^\omega \in L\}.$$

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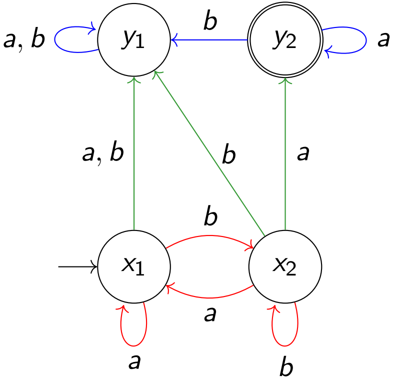
Idea: instead of working with a Büchi automaton that accepts L, work with an automaton that accepts $Lasso(L)$.

Automata for lasso languages

Definition (Ciancia, Venema, 2012)

A lasso automaton is a tuple $A = (\Sigma, X, Y, s, \rho, \sigma, \xi, F)$ with:

- finite alphabet Σ;
- two sets of states X, Y;
- three types of transitions:
  $\rho : X \times \Sigma \to X$,
  $\sigma : X \times \Sigma \to Y$,
  $\xi : Y \times \Sigma \to Y$;
- initial state $s \in X$;
- final states $F \subseteq Y$.



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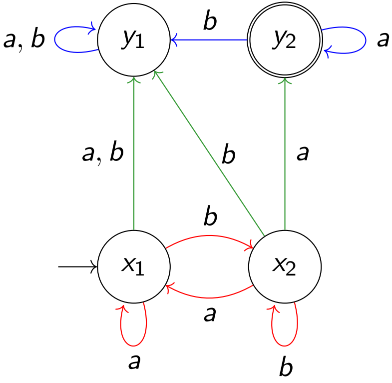

The lasso language of A is

$$Lasso(A) := \{(u, v) \mid s \xrightarrow{u}_{\rho} q \xrightarrow{v[0]}_{\sigma} q' \xrightarrow{v[1..]}_{\xi} q'' \in F\}.$$

Algebraic and coalgebraic view

A lasso automaton without final states is an algebra $(1 + \Sigma \times X, \Sigma \times X + \Sigma \times Y) \xrightarrow{s+\rho,\, \sigma+\xi} (X, Y)$ over Set $\times$ Set.
We call it an algebraic lasso automaton.

A lasso automaton without an initial state is a coalgebra $(X, Y) \xrightarrow{\rho\times\sigma,\, \xi\times F} (X^\Sigma \times Y^\Sigma, Y^\Sigma \times 2)$ over Set $\times$ Set.
We call it a coalgebraic lasso automaton.

Lasso automata for ω-regular languages

If L is ω-regular, then $Lasso(L)$ is saturated, i.e.:

$$u_1 v_1^\omega = u_2 v_2^\omega \implies \big((u_1, v_1) \in Lasso(L) \iff (u_2, v_2) \in Lasso(L)\big).$$

Definition (Ciancia, Venema, 2012)

An Ω-automaton is a lasso automaton A s.t. $Lasso(A)$ is saturated.

Theorem (Ciancia, Venema, 2012)

L is ω-regular iff $Lasso(L)$ is accepted by an Ω-automaton.

The algebraic approach

Wilke algebras are algebraic structures that recognise ω-regular languages.
Similar to semigroups, which recognise regular languages.

Definition (Wilke algebra (Wilke, 1993))

A Wilke algebra is of the form $W = (W^{fin}, W^{inf}, \cdot, \times, (-)^{\omega})$, where for all $s, t \in W^{fin}$ and $\alpha \in W^{inf}$:

- $s \cdot t \in W^{fin}$ (concatenation);
- $s \times \alpha \in W^{inf}$ (concatenation);
- $s^{\omega} \in W^{inf}$ (infinite power),

satisfying associativity, mixed associativity, circularity, coherence.

Wilke algebra recognition

A Wilke algebra homomorphism is a structure preserving map between Wilke algebras.

The Wilke algebra $(\Sigma^+, \Sigma^{\mathrm{UP}})$ is freely generated from Σ.

Definition

A quotient Wilke algebra is a triple (W^{fin}, W^{inf}, f), where:

- (W^{fin}, W^{inf}) is a Wilke algebra and
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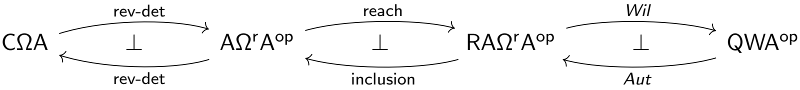
A quotient Wilke algebra (W^{fin}, W^{inf}, f) recognises a language L if $L = f^{-1}(U)$ for some $U \subseteq W^{inf}$.

Theorem (Wilke, 1993)

Given a language L, we have $L = UP(M)$ for some ω-regular M if and only if L is recognised by a finite quotient Wilke algebra.

Main result: a chain of adjunctions



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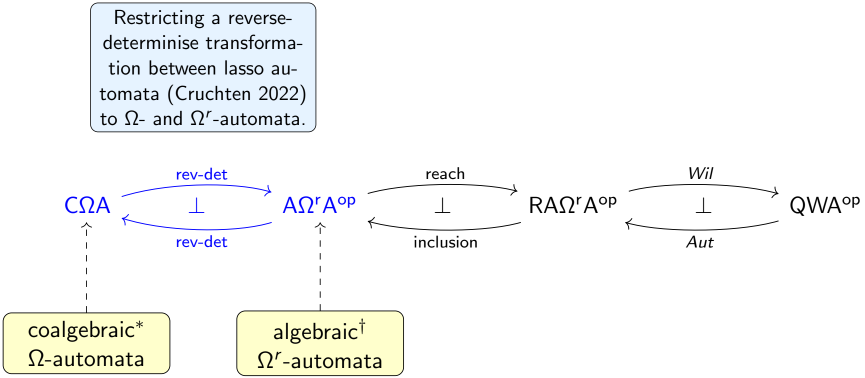

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(†) algebraic = no final states

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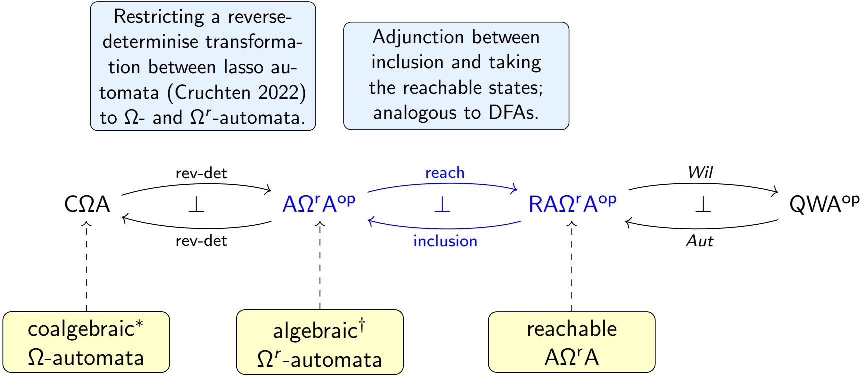

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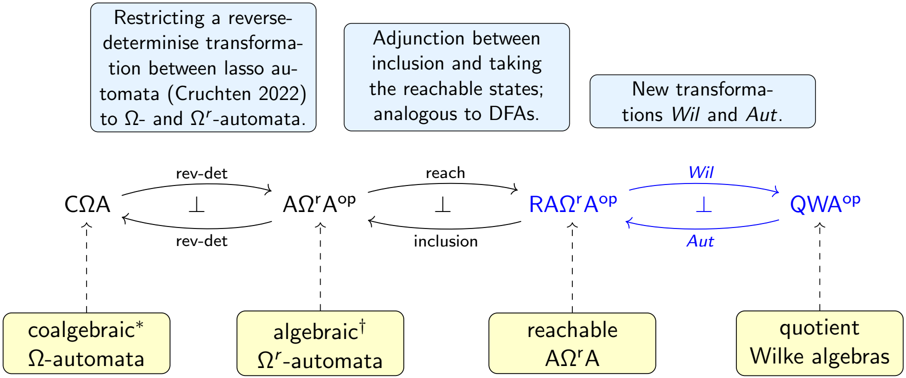

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Summary

ω-regular languages are characterised by:

- Büchi automata;
- Ω-automata (via lassos);
- quotient Wilke algebras (via ultimately periodic words).

Contribution: a dual adjunction between coalgebraic Ω-automata and quotient Wilke algebras.

Thank you!