

Categories of Kirchhoff Relations

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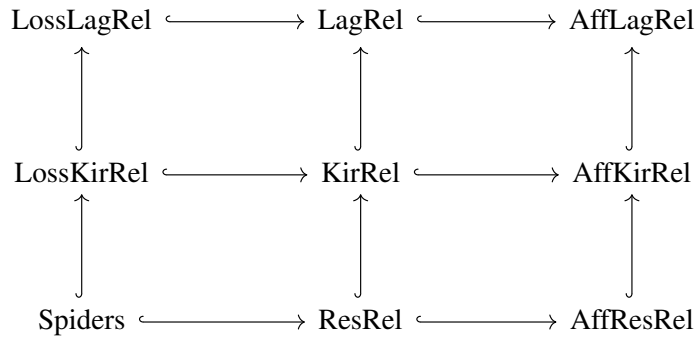
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1 Extended Abstract

It is known that the category of affine Lagrangian relations, AffLagRel_F , over the field, F , of integers modulo a prime p (with $p > 2$) is isomorphic to the category of stabilizer quantum circuits for p -dits [4]. Furthermore, it is known that electrical circuits (generalized for the field F), form a natural subcategory of AffLagRel_F [1, 3]. The purpose of this paper is to provide a characterization of this subcategory – and for important subcategories thereof – in terms of parity-check matrices as used in error detection.

A key subcategory, $\text{KirRel}_F \subseteq \text{LagRel}_F$, consists of those Lagrangian relations that satisfy Kirchhoff’s current law. The maps in this subcategory are generated by electrical components (generalized for the field F): namely junctions, resistors, and current dividers. Current dividers add an interesting quasi-stochastic aspect to the maps: when one insists that maps are deterministic one obtains the subcategory ResRel , which consists precisely of resistor circuits (generalized to F).

Below are displayed the categories which we characterize using parity-check matrices. The first column is the “lossless” circuits – those with zero output power – the last column is the affine categories.



Any Lagrangian relation can be converted into a state by precomposing with appropriate cups. Parity-check matrices for states in the category of Lagrangian relations have a standard form:

Theorem 1.1. *The parity-check matrix H for a Lagrangian subspace $\mathcal{R} \subseteq (F^n)^2$ can be put into the following standard form:*

$$H = \begin{pmatrix} Y & 0 & 1_{n_p} & A^T \\ -A & 1_{n_q} & 0 & 0 \end{pmatrix} \sigma_S \quad (1)$$

which is determined by the triple (A, Y, σ) where $n_p + n_q = n$ (the symplectic dimension of \mathcal{R}), $\sigma_S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ is a symplectic permutation, A has dimensions $n_q \times n_p$, and $Y = Y^T$.

We now define the subcategory of Lagrangian relations which we refer to as ‘‘Kirchhoff relations’’ similar categorical structures have been investigated in [2, 5]. Our approach using parity-check matrices, besides being novel, brings some new subcategories to the fore.

Definition 1.2. A Lagrangian relation $\mathcal{R} : n \rightarrow m$ in LagRel satisfies:

- (i) **Kirchhoff’s Current Law** if, for all $((q, p), (q', p')) \in \mathcal{R}$, the equality $\sum_{j=1}^n p_j = \sum_{k=1}^m p'_k$ holds
- (ii) **Translation invariance** if, whenever $\lambda \in F$ and $((q, p), (q', p')) \in \mathcal{R}$, then $((q + \vec{\lambda}_m, p), (q' + \vec{\lambda}_m, p')) \in \mathcal{R}$, where $\vec{\lambda}_n$ is a vector of dimension n all of whose components are the same $\lambda \in F$.

Kirchhoff’s current law is, for Lagrangian relations, equivalent to translation invariance:

Proposition 1.3. For a state \mathcal{R} in $\text{LagRel}(F)$, \mathcal{R} satisfies the Kirchhoff current law if and only if \mathcal{R} satisfies translational invariance.

The parity check matrix for a Kirchhoff relations is given by the following proposition:

Proposition 1.4. A Lagrangian relation is a Kirchhoff relation if and only if every standard parity-check matrix has $Y\vec{1} = 0$ and A quasi-stochastic, that is $A\vec{1} = \vec{1}$.

The Kirchhoff relations form a category, KirRel , which, as electrical circuits, is generated by current dividers, junctions, and resistors. A further subcategory of ‘‘deterministic’’ Kirchhoff relations, ResRel , can be defined which corresponds precisely to resistor circuits:

Definition 1.5. A deterministic Kirchhoff relation $\mathcal{R} : n \rightarrow m$ is a Kirchhoff relation which additionally satisfies the following property:

- There is an equivalence relation \sim on $\underline{n+m} = \{1, \dots, n+m\}$,
- For each $i \in \underline{n+m}$, there is an $\begin{pmatrix} q \\ p \end{pmatrix} \in \mathcal{R}$ with $q_j = 1$ when $i \sim j$ and $q_j = 0$ when $j \not\sim i$ (thus, the equivalence classes can be separated by the momenta).

Using this definition one can prove the following proposition:

Proposition 1.6. For any $\mathcal{R} \in \text{LagRel}$:

- (i) \mathcal{R} is deterministic Kirchhoff if and only if, in every standard parity check matrix, the submatrix A is (not only quasi-stochastic but also) deterministic.
- (ii) Deterministic relations form subprop $\text{ResRel} \subseteq \text{LagRel}$, in particular, deterministic Kirchhoff relations are closed to composition.

Link to full paper

Title of paper: Categories of Kirchhoff relations

Identifier: submit/4303229

Link to ArXiv preprint: <https://arxiv.org/pdf/2205.05870.pdf>

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