Categories of Kirchhoff Relations

Robin Cockett
University of Calgary
Alberta, Canada
Department of Computer Science
robin@ucalgary.ca

Amolak Ratan Kalra
University of Calgary
Alberta, Canada
Department of Computer Science
amolakratan.kalra@ucalgary.ca

Shiroman Prakash
Dayalbagh Educational Institute
Uttar Pradesh, India
Department of Physics and Computer Science
sprakash@dei.ac.in

1 Extended Abstract

It is known that the category of affine Lagrangian relations, $\mathsf{AffLagRel}_\mathsf{F}$, over the field, F, of integers modulo a prime p (with $p > 2$) is isomorphic to the category of stabilizer quantum circuits for p-dits [4]. Furthermore, it is known that electrical circuits (generalized for the field F), form a natural subcategory of $\mathsf{AffLagRel}_\mathsf{F}$ [1,3]. The purpose of this paper is to provide a characterization of this subcategory – and for important subcategories thereof – in terms of parity-check matrices as used in error detection.

A key subcategory, $\mathsf{KirRel}_F \subseteq \mathsf{LagRel}_F$, consists of those Lagrangian relations that satisfy Kirchhoff's current law. The maps in this subcategory are generated by electrical components (generalized for the field F): namely junctions, resistors, and current dividers. Current dividers add an interesting quasi-stochastic aspect to the maps: when one insists that maps are deterministic one obtains the subcategory ResRel, which consists precisely of resistor circuits (generalized to F).

Below are displayed the categories which we characterize using parity-check matrices. The first column is the "lossless" circuits – those with zero output power – the last column is the affine categories.

$$\begin{array}{ccccc}
\text{LossLagRel} & \hookrightarrow & \text{LagRel} & \hookrightarrow & \text{AffLagRel} \\
\uparrow & & \uparrow & & \uparrow \\
\text{LossKirRel} & \hookrightarrow & \text{KirRel} & \hookrightarrow & \text{AffKirRel} \\
\uparrow & & \uparrow & & \uparrow \\
\text{Spiders} & \hookrightarrow & \text{ResRel} & \hookrightarrow & \text{AffResRel}
\end{array}$$

Any Lagrangian relation can be converted into a state by precomposing with appropriate cups. Parity-check matrices for states in the category of Lagrangian relations have a standard form:

Theorem 1.1. *The parity-check matrix H for a Lagrangian subspace $\mathscr{R} \subseteq (F^n)^2$ can be put into the following standard form:*

$$H = \begin{pmatrix} Y & 0 & 1_{n_p} & A^T \\ -A & 1_{n_q} & 0 & 0 \end{pmatrix} \sigma_S \tag{1}$$

Submitted to:
ACT 2022

which is determined by the triple (A, Y, σ) *where* $n_p + n_q = n$ *(the symplectic dimension of* $\mathscr{R}$*),* $\sigma_S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ *is a symplectic permutation, A has dimensions* $n_q \times n_p$*, and* $Y = Y^T$.

We now define the subcategory of Lagrangian relations which we refer to as "Kirchhoff relations" similar categorical structures have been investigated in [2, 5]. Our approach using parity-check matrices, besides being novel, brings some new subcategories to the fore.

Definition 1.2. *A Lagrangian relation* $\mathscr{R} : n \to m$ *in* LagRel *satisfies:*

(i) ***Kirchhoff's Current Law*** *if, for all* $((q,p),(q',p')) \in \mathscr{R}$*, the equality* $\sum_{j=1}^{n} p_j = \sum_{k=1}^{m} p'_k$ *holds*

(ii) ***Translation invariance*** *if, whenever* $\lambda \in F$ *and* $((q,p),(q',p')) \in \mathscr{R}$*, then* $((q+\vec{\lambda}_m, p),(q'+\vec{\lambda}_m, p'_n)) \in \mathscr{R}$*, where* $\vec{\lambda}_n$ *is a vector of dimension n all of whose components are the same* $\lambda \in F$.

Kirchhoff's current law is, for Lagrangian relations, equivalent to translation invariance:

Proposition 1.3. *For a state* $\mathscr{R}$ *in* LagRel(F)*,* $\mathscr{R}$ *satisfies the Kirchhoff current law if and only if* $\mathscr{R}$ *satisfies translational invariance.*

The parity check matrix for a Kirchhoff relations is given by the following proposition:

Proposition 1.4. *A Lagrangian relation is a Kirchhoff relation if and only if every standard parity-check matrix has* $Y\vec{1} = 0$ *and A quasi-stochastic, that is* $A\vec{1} = \vec{1}$.

The Kirchhoff relations form a category, KirRel, which, as electrical circuits, is generated by current dividers, junctions, and resistors. A further subcategory of "deterministic" Kirchhoff relations, ResRel, can be defined which corresponds precisely to resistor circuits:

Definition 1.5. *A deterministic Kirchhoff relation* $\mathscr{R} : n \to m$ *is a Kirchhoff relation which additionally satisfies the following property:*

- *There is an equivalence relation* $\_ \sim \_$ *on* $\underline{n+m} = \{1, ..., n+m\}$,
- *For each* $i \in \underline{n+m}$*, there is an* $\begin{pmatrix} q \\ p \end{pmatrix} \in \mathscr{R}$ *with* $q_j = 1$ *when* $i \sim j$ *and* $q_j = 0$ *when* $j \not\sim i$ *(thus, the equivalence classes can be separated by the momenta).*

Using this definition one can prove the following proposition:

Proposition 1.6. *For any* $\mathscr{R} \in$ LagRel*:*

(i) $\mathscr{R}$ *is deterministic Kirchhoff if and only if, in every standard parity check matrix, the submatrix A is (not only quasi-stochastic but also) deterministic.*

(ii) Deterministic relations form subprop ResRel $\subseteq$ LagRel*, in particular, deterministic Kirchhoff relations are closed to composition.*

Link to full paper
Title of paper: Categories of Kirchhoff relations
Identifier: submit/4303229
Link to ArXiv preprint: https://arxiv.org/pdf/2205.05870.pdf

References

[1] John C Baez, Brandon Coya & Franciscus Rebro (2017): *Props in network theory*. *arXiv preprint arXiv:1707.08321*.

[2] John C Baez & Brendan Fong (2015): *A compositional framework for passive linear networks*. *arXiv preprint arXiv:1504.05625*.

[3] Filippo Bonchi, Robin Piedeleu, Pawel Sobocinski & Fabio Zanasi (2019): *Graphical Affine Algebra*. *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 1–12.

[4] Cole Comfort & Aleks Kissinger (2021): *A Graphical Calculus for Lagrangian Relations*.

[5] Brendan Fong (2016): *The algebra of open and interconnected systems*. *arXiv preprint arXiv:1609.05382*.