Extended Abstract: Dynamical Systems via Domains

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Topic Famously, domain theory provided a mathematical semantics for symbolic computation: systematically explaining code of a programming language by assigning it to the mathematical object computed by it. We lack this kind of explainability for the non-symbolic computation performed, e.g., by neural networks in artificial intelligence. This talk, which is based on my PhD thesis [4, ch. 4–5], suggests an extension of domain-theoretic semantics to non-symbolic computation.

Generally, non-symbolic computation can be viewed as a dynamical system: a set of (computational) states together with a dynamics describing how to move from one state to the next (program). Thus, the task is to construct, for a given dynamical system, a 'domain' that describes the behavior of the system.[1] Domains, i.e., the object of study of domain theory, are certain partial orders whose elements intuitively are regarded as outputs of computational processes while the order describes information containment.

To build the semantics, the high-level idea is this: Given a dynamical system $\mathfrak{X}$, we construct a structure $\mathfrak{D}$ that we will call the *dynamical domain* of $\mathfrak{X}$. Intuitively, $\mathfrak{D}$ consists of 'basic' elements that represent increasingly finer observations of the system $\mathfrak{X}$ together with the 'limits' of these basic elements. The additional domain-theoretic structure on $\mathfrak{D}$ is such that it induces a dynamical system on these limit elements, which we call the *system modeled by* $\mathfrak{D}$. We show that this modeled system is isomorphic to $\mathfrak{X}$. Categorically speaking, we then prove that the construction of the dynamical domain for a system and the construction of the system modeled by a dynamical domain are adjoint. This allows studying dynamical systems through the rich domain theory. We now explain this construction.

Contribution There are many (formal) notions of dynamical systems. Here we work with a very general one: A *dynamical system* is a structure $\mathfrak{X} = (X, \mathscr{A}, \mu, T)$ where the state space $(X, \mathscr{A}, \mu)$ is a probability space (assumed to be standard Borel) and the dynamics $T : X \to X$ is measurable. Often, one assumes the probability space to be Lebesgue and T to be a measure-preserving bijection; we then call $\mathfrak{X}$ *standard*.

In computational applications, standardness seems too strong, hence the general definition. An example is machine learning: At each training stage, the machine is characterized by a set w of parameters (or *w*eights). Given some *d*ata d, the *l*earning algorithm (e.g., backpropagation) produces a new set of parameters $w' = L(w,d)$. We can view this as a dynamical system: The state space $X = W \times D^{\omega}$ is the space of parameters W times the space of sequences of data D^{ω}. The dynamics $T : X \to X$ is defined by mapping (w, δ) to $\big(L(w, \delta_0)\ ,\ (\delta_1, \delta_2, \dots)\big)$. If a probability measure on X describes the random initialization of the machine and the data sampling, there is no reason to expect T to be measure-preserving.

Now, given a dynamical system $\mathfrak{X} = (X, \mathscr{A}, \mu, T)$, the dynamical domain $\mathfrak{D} = (D, v, f)$ is constructed as a limit of finite domain-theoretic structures as described in the following paragraphs.

[1] If preferd, one can omit the terminology of 'symbolic/non-symbolic computation', and just talk about dynamical systems.

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A measurable subset A of X can be regarded as an observation or measurement that we can make about the system: if the system is in a state $x \in A$, making measurement A comes out positive. So if we have a finite set $\mathscr{C}$ of measurable sets that cover the state space X, it provides a finite and non-deterministic dynamical system that 'reflects' the original system: the states are the elements from $\mathscr{C}$ and there is a connection from state A to state B if there is $x \in A$ with $T(x) \in B$. For $n \geq 0$, call $i = (n, \mathscr{C})$ an observation parameter (n intuitively describes for how many steps we observe the system through $\mathscr{C}$). A state $x \in X$ induces the set $\mathscr{O}_i(x)$ of trajectories in this observed system: namely those $t = (A_0, A_1, \ldots, A_{n-1})$ such that $T^k(x) \in A_k$ (for $k = 0, \ldots, n-1$). The domain-theoretic concept known as the Smyth powerdomain suggests considering the set D_i of nonempty finite collections of such $\mathscr{O}_i(x)$ ordered by reverse inclusion. It has the monotone function $f_i(M) = \{\mathscr{O}_i(T(y)) : \mathscr{O}_i(y) \in M\}$. And a valuation v_i induced by assigning each $\mathscr{O}_i(x)$ the value $\mu\big\{y \in X : \mathscr{O}_i(y) = \mathscr{O}_i(x)\big\}$.

Thus, we obtain the structure $\mathfrak{D}_i = (D_i, v_i, f_i)$, where D_i is known as a finite Scott domain, v_i as a valuation, and $f_i : D_i \to D_i$ as a Scott-continuous function. Now, by considering a cover $\mathscr{D}$ that refines $\mathscr{C}$ and $m \geq n$, we get a 'refined' observation parameter $j = (m, \mathscr{D})$ and a natural map $p_{ij} : D_j \to D_i$. So we have a diagram $(\mathfrak{D}_i, p_{ij})$ and want to take the limit to get the dynamical domain that models system $\mathfrak{X}$.

The category in which we build the limit will be the category of dynamical domains. We define it independently in a common domain-theoretic way. First, we specify a collection of finite domains, and then define the desired category to consist of those objects that are obtained as appropriate limits of appropriate diagrams of finite domains. Here the finite domains are the structures $\mathfrak{D} = (D, v, f)$ where, as above, D is a finite Scott domain, v is a valuation on D, and $f : D \to D$ is Scott-continuous function. But, for reasons of space, we need to skip the definitions of morphism [4, def. 4.4.2] and 'appropriate diagram' [4, def. 4.4.7].[2] The main conceptual contribution is to show that the appropriate limit $\mathfrak{D} = (D, v, f)$ of an appropriate diagram $(\mathfrak{D}_i, p_{ij})_I$ does indeed exist. To construct D and v, one can use existing work [1, 3], but the difficulty is with f. We show that there is a canonical choice for f if it has to map maximal elements of D to maximal ones. And we want this since then $\mathfrak{D}$ models the dynamical system whose state space $\max D$ consists of the maximal elements of D, with a probability measure induced by v, and the dynamics $f : \max D \to \max D$. By adding further conditions, we can also define standard dynamical domains and get that the dynamical systems modeled by them are standard.

The main results then are:

(1) For every (standard) dynamical system $\mathfrak{X}$, there is a (standard) dynamical domain $\mathfrak{D}$—constructed by observing the system $\mathfrak{X}$ as described above—such that the (standard) dynamical system modeled by $\mathfrak{D}$ is metrically isomorphic to $\mathfrak{X}$.

(2) The construction of the dynamical domain for a system and the construction of the system modeled by a dynamical domain are adjoint.

A corollary of (1) is that every system can be realized as one over a compact space with a continuous dynamics (a topological representation result, similar to the Jewett–Krieger theorem [5, sec. 4.4]). An extension of (2) is that the category of dynamical systems is a localization of the category of certain topological systems which, in turn, is equivalent to an interesting subcategory of dynamical domains.[3]

[2] To at least state the former, a morphism $\alpha : (D, v, f) \to (D', v', f')$ is a Scott-continuous function $\alpha : D \to D'$ such that (a) $f(\max D) \subseteq \max D'$, where $\max D$ are the maximal elements of D, (b) for $a \in D$ and $y \in \max D'$ with $\alpha(a) \leq y$, there is $a \leq x \in \max D$ with $\alpha(x) = y$, (c) for a Scott-open set $U \subseteq D'$, $v'(U) = v(\alpha^{-1}(U))$, and (d) for $x \in \max D$, $\alpha(f(x)) \geq f'(\alpha(x))$.

[3] Inspiring contributions from an applied category theory perspective to the present broader topic of dynamical systems, computation, and machine learning are, e.g., [2, 6].

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