STRONG SHIFT EQUIVALENCE AND BIALGEBRAS IN TRACED MONOIDAL CATEGORIES

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Abstract. In this paper, we present a new way to investigate the main problem of symbolic dynamics, the conjugacy problem, by proving that this problem actually relates to a natural question in category theory regarding the theory of bialgebras with a trace.

As a consequence of this theory, we obtain a systematic way of obtaining new invariants for the conjugacy problem by looking at existing bialgebras in the literature.

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Introduction

Symbolic dynamics [4] was developed in the late 30s as a tool for analyzing dynamical systems by discretizing the space, representing trajectories as biinfinite paths in a (hopefully finite) graph. Possibly the most important question in symbolic dynamics is the decidability of the conjugacy problem: decide if two symbolic dynamic systems (more precisely subshifts of finite type) are isomorphic [2].

The conjugacy problem can be easily reformulated in terms of matrices. Given two matrices of nonnegative integers, M and N we say that M, N are *strong shift equivalent* (represent isomorphic dynamical systems) if and only if $M \equiv N$ where $\equiv$ is the smallest equivalence relation s.t. $RS \equiv SR$ for all nonnegative, possibly nonsquare, matrices R, S.

For instance, the matrices $M = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 4 & 3 \\ 8 & 9 & 5 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 7 \\ 5 & 8 \end{pmatrix}$ are strong shift equivalent. Indeed $M = \begin{pmatrix} 0 & 2 \\ 1 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 3 \\ 4 & 5 \end{pmatrix}$.

Deciding conjugacy (strong shift equivalence) is open since nearly one century. Some criteria exists to show some matrices are *not* equivalent, but some annoyingly-simple examples are still out of reach. For instance, it is conjectured that $\begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 12 \\ 1 & 1 \end{pmatrix}$ are equivalent, but no proof is known.

To investigate this problem, we propose in this article to use category theory. The main results of the paper is that there is a strong link between strong shift equivalence, and (bicommutative) bialgebras in traced monoidal categories.

More precisely, consider the prop P given by the following generators and relations: We have one commutative monoid and one commutative comonoid that form a bialgebra, we have a map which is both a morphism for the monoid and comonoid, and the category has a trace. Then in this prop, the scalars can naturally be interpreted as the equivalence classes for strong shift equivalence.

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From the categorical point of view, the foundation for this theorem is the basic idea that the bialgebra prop is exactly the prop of nonnegative integer matrices, probably observed first in [5]. As symbolic dynamics is interested in a particular equivalence relation on nonnegative integer matrices, this is a good start to think about them categorically. As the equation $RS \equiv SR$ reminds of the trace, it is therefore not surprising that there is a strong link between the traced bialgebra prop P (matrices with a notion of traces) and strong shift equivalence.

The correspondence we obtain this way is however not the one we need. It reminds however of a work by David Hillman [3] between commutative algebras and *flow equivalence*, another equivalence notion on matrices coming from symbolic dynamics.

To go from flow equivalence to strong shift equivalence, one needs to work on the symbolic dynamics side, to show that strong shift equivalence can be defined as a generalization of flow equivalence to matrices with coefficients in $\mathbb{Z}_+[t]$.

This categorical manipulation of the concepts of symbolic dynamics is not just an intellectual game but serves some purpose, as it has actually the following consequence: Suppose one knows a category C which contains a traced bialgebra (many examples abound in the literature, the most prominent examples being finite dimensional Hopf algebras). The universal property implies that there is a functor from the traced bialgebra prop P to C. This implies one can interpret matrices in the category C in such a way that two matrices that are strong shift equivalent will have the same interpretation.

This concept is known as an *invariant* in symbolic dynamics: a map f s.t. $f(M) = f(N)$ if $M \equiv N$. As the equivalence problem is not known to be decidable, invariants offer an alternative to prove that two matrices are not equivalent: just find an invariant f s.t. $f(M) \neq f(N)$.

What is interesting with the categorical approach is that we recover, by examining well-known examples of bialgebras, all important examples of invariants in the symbolic dynamics literature.

One example is the binomial bialgebra $\mathbb{R}[X]$ with monoid $m(X^n \otimes X^m) = X^{n+m}$ and comonoid $\Delta(X^n) = \sum_k \binom{n}{k} X^k \otimes X^{n-k}$. The corresponding category C does not come with a trace, but we can change it slightly (switching from $\mathbb{R}$ to nonnegative, possibly infinite, reals) to obtain a trace. Taking $h(X^n) = \lambda^n X^n$ as our choice of morphism for both the monoid and comonoid, we obtain from the universal property a functor from P to C, which on matrices, correspond to one of the most well-known invariants of symbolic dynamics, the Zeta function [1] $\zeta_M(t) = 1/\det(I - tM)$ evaluated at λ.

We also have the added bonus that our approach is complete: there exist at least one category for which having the same interpretation is a necessary and sufficient condition for strong shift equivalence.

In the full version of the article, we start by giving the classical definitions of a prop, then we introduce traced props, and show the main theorem: the traced completion of matrices with noninteger coefficients corresponds to matrices quotiented by flow equivalence, and the traced completion of matrices with coefficients in $\mathbb{Z}_+[t]$ corresponds to matrices quotiented by strong shift equivalence.

In the last section, we explain how we can recover classical and new invariants for symbolic dynamics by exploiting the idea of interpreting matrices in traced bialgebra props. This section is developed from the point of view of someone who knows no invariant of strong shift equivalence: we look at existing and well known bialgebras and see which invariant we get from them. The fact one obtains this way fairly well known invariants from symbolic dynamics acts as a proof that the whole approach is successful.

While the perfect reader for this article is someone familiar both with category theory (esp. the categorical approach to universal algebra developed by Lawvere) and symbolic dynamics, it should be accessible however without any knowledge of symbolic dynamics.

References

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