Characterization of Contextuality with Semi-Module Čech Cohomology

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I present a generalized notion of obstruction in Čech cohomology on semi-modules, which allows one to characterize non-disturbing contextual behaviors when they have distributions in any semi-field, generalizing the usual Čech cohomology used in the sheaf approach to contextuality.

Sheaf approach The sheaf approach to contextuality [_2012_2012], a well-known kind of non-classicality, has as fundamental objects the measurements X and their contexts $\mathscr{U}$ given by their compatibility, defining a site with the inclusion map. A measurement scenario is given by the sheaf of events $\mathscr{E} : \langle X, \mathscr{U} \rangle^{op} \to$ **Set**, attaching the sets of outcomes O^U for each context $U \in \mathscr{U}$. R-empirical models are defined with a semi-ring R, usually the probability $\mathbb{R}^+$ and the Boolean $\mathbb{B}$, by a functor $\mathscr{D}_R : \mathbf{Set} \to \mathbf{Set} :: O^U \mapsto \left\{ \mu_R^{O^U} \right\}$ on a measurement scenario, taking a set of local events to the set of R-measures defined on it $\mu_p^{O^U} : \mathbb{P}\left(O^U\right) \to R$ that satisfies $\mu_R^{O^U}(O^U) = 1_R$. The non-disturbance condition is usually imposed by saying that $\mu_R^{O^j}|_{kj} = \mu_R^{O^k}|_{kj}$ for all k and j. By imposing the independence of hidden variables λ on the context called λ-independence, one can write

$$\mu_R^{O^U}(A) = \sum_{\Lambda} p(\lambda) \prod_{x \in U} \mu_R^{O^x}(\rho'(U,x)(A)), \tag{1}$$

with $p(\Lambda) = 1_R$. An R-empirical model is said R-non-contextual if there is a R-measure p and a set of hidden variables Λ such that Equation 1 holds for all $U \in \mathscr{U}$. We get the Fine-Abramsky-Brandenburguer theorem by imposing $\prod_{x \in U} \mu_R^{O^x} \in \{0_R, 1_R\}$ (outcome-determinism), that identifies Λ with the set of global events. Therefore, contextuality is the failure to extend a local section to a global section of $\mathscr{D}_R \mathscr{E}$.

Čech cohomology We define the nerve $\mathrm{N}(\mathscr{U})$ of maximal contexts, which defines an abstract simplicial complex, where the q-simplices form a collection $\mathrm{N}(\mathscr{U})^q$, and with it the boundary map. Defining a functor $\mathscr{F} : \mathrm{N}(\mathscr{U}) \to \mathbf{AbGrp}$ related to $\mathscr{D}_R \mathscr{E}$, typically $\mathscr{F} = F_S \mathscr{L}$ with $\mathscr{L}$ at least a subsheaf of $\mathscr{E}$ and $F_S : \mathbf{Set} \to \mathbf{AbGrp}$ that assigns to a set O the free Abelian group $F_S(O)$ generated by it related to a ring S, plus some properties, we get an augmented Čech cochain complex

$$\mathbf{0} \longrightarrow C^0(\mathscr{U}, \mathscr{F}) \xrightarrow{d^0} C^1(\mathscr{U}, \mathscr{F}) \xrightarrow{d^1} C^2(\mathscr{U}, \mathscr{F}) \xrightarrow{d^2} \dots \tag{2}$$

and the coboundary maps as the group homomorphisms satisfying $d^{q+1} d^q = 0$, thus we can construct the Čech cohomology $\check{H}^q(\mathscr{U}, \mathscr{F})$ of this cochain complex. With the identification of $\check{H}^0(\mathscr{U}, \mathscr{F})$ and the set of compatible families, one can characterize cohomological contextuality by the construction of obstruction on an initial 0-cochain that codifies the local sections of $\mathscr{F}$ on the covering $\mathscr{U}$. The key elements are the relative cohomology $\mathscr{F}_{\tilde{U}}$ and the obstruction $\gamma(s_{j_0})$. One can prove:

Submitted to:
ACT 2022

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Proposition 1. *Let $\mathscr{U}$ be connected, $U_{j_0} \in \mathscr{U}$, and $s_{j_0} \in \mathscr{F}(U_{j_0})$. Then $\gamma(s_{j_0}) = 0$ if and only if there is a compatible family $\left\{r_{j_k} \in \mathscr{F}(U_{j_k})\right\}_{U_{j_k} \in \mathscr{U}}$ such that $r_{j_0} = s_{j_0}$.*

If an obstruction is non-trivial, then the empirical model must be contextual. However, the representation via groups has a cost. The functor $F : \mathbf{Ring} \to \mathbf{Rig}$ to the category of semi-rings forgets property and structure, which allows the violation of the cohomological characterization of contextual behavior.

Čech cohomology on semi-modules The objective of Ref. [3] is to characterize contextual behavior by keeping the semi-module on the original semi-ring, but when doing so, a lot of the properties of usual cohomology fail, once it is impossible to define the coboundary operators. But one can define a cochain complex of R-semi-modules consisting of R-semi-modules C^q and R-homomorphisms d_+^q, d_-^q [2]:

$$C = \ldots \underset{d_-^{q-2}}{\overset{d_+^{q-2}}{\rightrightarrows}} C^{q-2} \underset{d_-^{q-1}}{\overset{d_+^{q-1}}{\rightrightarrows}} C^q \underset{d_-^{q}}{\overset{d_+^{q}}{\rightrightarrows}} C^{q+1} \underset{d_-^{q+1}}{\overset{d_+^{q+1}}{\rightrightarrows}} \ldots \tag{3}$$

satisfying some imposed properties that generalize usual conditions. The cohomology R-semi-module $H^q(C) = Z^q(C)/\rho^q$ will depend on a non-unique congruence relation ρ^q in $Z^q(C)$. The Čech cohomology on R-semi-modules is defined as usual, but one needs to work with the differentials separately and one substitutes the presheaf $\mathscr{F}$ by a presheaf of R-semi-modules $\mathscr{G}$. The coboundary operators can be defined, and I choose $\mathscr{G}$ as a free R-semi-module generated by the local events for each element of $\mathscr{U}$ and intersections. We need a formalism that allows the construction of obstructions without subtraction. By allowing R to be a semi-field, we can use R-stochastic operators to codify the difference between the coboundaries. The difference operator is a function $[g_q] :: c \in C^q(\mathscr{U}, \mathscr{G}) \mapsto [g_q, c]$, defining for each cochain its unique class of difference cochains. Diagrammatically

$$\begin{array}{ccccccccc} & & C^1(\mathscr{U},\mathscr{G}) & & & & C^2(\mathscr{U},\mathscr{G}) & & \\ & \overset{d_+^0}{\nearrow} & \big\downarrow [g_0] & \overset{id}{\searrow} & & \overset{d_+^0}{\nearrow} & \big\downarrow [g_1] & \overset{id}{\searrow} & \\ 0 \longrightarrow C^0(\mathscr{U},\mathscr{G}) & & & & C^1(\mathscr{U},\mathscr{G}) & & & & C^2(\mathscr{U},\mathscr{G}) \quad \cdots \\ & \underset{d_-^0}{\searrow} & & \underset{id}{\nearrow} & & \underset{d_-^0}{\searrow} & & \underset{id}{\nearrow} & \\ & & C^1(\mathscr{U},\mathscr{G}) & & & & C^2(\mathscr{U},\mathscr{G}) & & \end{array} \tag{4}$$

Theorem 2. *Let $\mathscr{U}$ be connected, $U_{j_0} \in \mathscr{U}$, and $c_{j_0} \in \mathscr{G}(U_{j_0})$. Then $\gamma(c_{j_0})$ is trivial if and only if there is a compatible family $\left\{r_{j_k} \in \mathscr{G}(U_{j_k})\right\}_{U_{j_k} \in \mathscr{U}}$ such that $c_{j_0} = r_{j_0}$.*

Corollary 3. *A model is R-contextual if and only if there is a local section s_i representing a local event by $\mathscr{G}$ with non-trivial obstruction.*

A similar result already exists in the level of effect algebras, which directly compares to a Boolean effect algebra via extendability. The two approaches search for the same thing: to verify if a structure can be understood as a Boolean structure, if it can be understood classically in deterministic ways.

References

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