A Diagrammatic View of Differential Equations in Physics

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This is an extended abstract for a preprint by the present authors [11].

Abstract. Presenting systems of differential equations in the form of diagrams has become common in certain parts of physics, especially electromagnetism and computational physics. In this work, we aim to put such use of diagrams on a firm mathematical footing, while also systematizing a broadly applicable framework to reason formally about systems of equations and their solutions. Our main mathematical tools are category-theoretic diagrams, which are well known, and morphisms between diagrams, which have been less appreciated. As an application of the diagrammatic framework, we show how complex, multiphysical systems can be modularly constructed from basic physical principles. A wealth of examples, drawn from electromagnetism, transport phenomena, fluid mechanics, and other fields, is included.

1 Context

Diagrammatic presentations of physical theories have been studied by many authors in both the physical sciences and in mathematics [2, 3, 4, 6, 9, 15]. Loosely speaking, such diagrams are directed graphs, where the nodes represent physical quantities (such as fields or densities) and the arrows represent equations between the physical quantities. More precisely, a single arrow $f\colon x \to y$ asserts the equation $f(x) = y$ constraining the physical quantities x and y; multiple arrows $f\colon x \to y$ and $g\colon z \to y$ pointing to the same node represent a sum of values, asserting the equation $f(x) + g(z) = y$. However, even with these two simple conventions, many diagrams in the literature suffer from internal inconsistency, sometimes using multiple arrows with the same target to instead denote "parallel" equations (in the case above, $f(x) = y$ *and* $g(z) = y$).

This is just the tip of the iceberg, and it is an unsurprising consequence of the fact that these diagrams are not formal mathematical objects. Being informal is not a problem if we only wish to use the diagrams to guide our intuition or to help us understand the general landscape of a physical theory, but it *does* inhibit us from using them to reason precisely, as we prefer to do in mathematics. Indeed, one of the main proponents of diagrammatic presentations in physics was Enzo Tonti (whence the alternative name of "Tonti diagrams"), who aimed to use them to build a systematic classification of physical theories [16]. Although Tonti managed to avoid some of the inconsistencies present in other authors' diagrams, he still did not provide a mathematical definition of the objects in question. It was observed fairly early on that Tonti's diagrams resemble commutative diagrams as we know them in category theory [9], but this was never made precise.

The main goal of our article [11] is to finally make this analogy entirely precise, and exploit it to develop rigorous, compositional methods for presenting physical theories.

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2 Contributions

Using the 2-categorical notion of diagram categories [8, 7, 13, 12], we construct a mathematical framework for not only Tonti diagrams but also the morphisms between them. We show that morphisms of diagrams, not previously considered in the physics literature on diagrammatic presentations, can be used to formulate initial value and boundary value problems. A wide variety of physical systems, from electromagnetism to transport phenomena to fluid mechanics, are all shown to fit into this framework, as well as non-differential examples, such as finite difference equations.

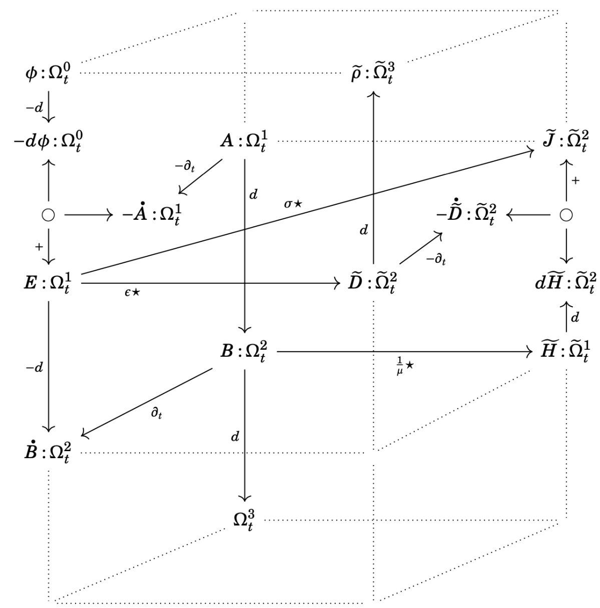

Figure 1: “Maxwell’s house”, presenting Maxwell’s equations in matter.

Several applications of the diagrammatic formalism are also developed in the paper. Using structured cospans [5, 1] and undirected wiring diagrams [14, 10], we show how to modularly combine distinct physical theories in order to compositionally construct multiphysics systems. Another use of morphisms is that they allow us to formally describe how different diagrams relate to one another, and to consider when two diagrams are “the same” — a question that ultimately leads to the philosophy of science, but for which we can give at least a partial answer: we define a notion of weak equivalence of diagrams, and show that weakly equivalent diagrams have solution sets that are in bijection with one another.

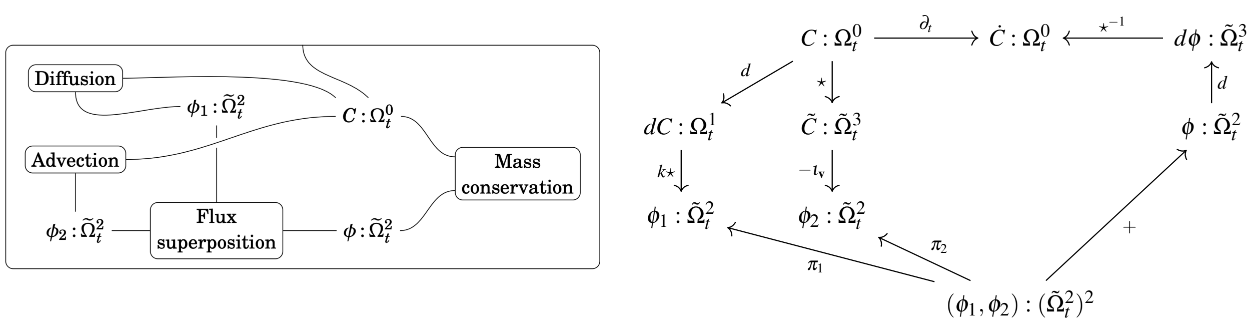

Figure 2: *Left:* the undirected wiring diagram (UWD) encoding the compositional structure of the advection-diffusion equation; *Right:* the diagram resulting from applying the UWD to relevant open cartesian diagrams.

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