A category-theoretic proof of the ergodic decomposition theorem

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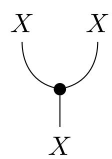
The ergodic decomposition theorem is a cornerstone result of dynamical systems and ergodic theory. It states that every invariant measure on a dynamical system is a mixture of ergodic ones. (See [4] for an expository, structural treatment of the topic.) Here we formulate and prove the theorem in terms of string diagrams, using the formalism of Markov categories. We recover the usual measure-theoretic statement by instantiating our result in the category of stochastic kernels. Along the way we give a conceptual treatment of several concepts in the theory of deterministic and stochastic dynamical systems. In particular,

- ergodic measures appear very naturally as particular cones of deterministic morphisms (in the sense of Markov categories);
- the invariant σ-algebra of a dynamical system can be seen as a colimit in the category of Markov kernels.

In line with other uses of category theory, once the necessary structures are in place, our proof of the main theorem is much simpler than traditional approaches. In particular, it does not use any quantitative limiting arguments. We hope that this result paves the way for further applications of category theory to dynamical systems, ergodic theory, and information theory.

Background. Markov categories have emerged in the last few years as a categorical framework for doing probability theory synthetically [3]. A *Markov category* is a symmetric monoidal category where each object X is equipped with canonical maps

copy = 

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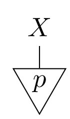
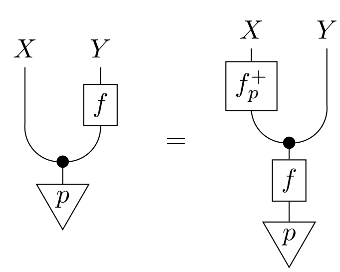
(read from bottom to top) making X a commutative comonoid. The morphisms are required to commute with the counits, but not with the comultiplications. The main Markov category of interest for us is the category Stoch of measurable spaces and stochastic (Markov) kernels. We refer to [3] for further examples.

In a Markov category, a *state* of an object X is a morphism from the monoidal unit $m : I \to X$, which we denote as follows.



In Stoch, this is precisely a measure on X. A morphism $f : X \to Y$ in a Markov category is called *deterministic* if it commutes with copying. In Stoch, deterministic states are precisely the *zero-one measures*, i.e. those probability measures that assign to each event probability either 0 or 1. Canonical examples of these are Dirac deltas, but those are not the only examples (see [2] for more). As we show below, ergodic measures are examples too.

Let us say that an object X in a Markov category is *regular* if for every object Y, every state $p : I \to X$ and every deterministic morphism $f : X \to Y$ there exists a *Bayesian inverse* $f_p^+ : Y \to X$ such that the following Bayes-type equation holds.



In Stoch, all standard Borel spaces (also called 'regular measurable spaces') are regular in this sense. Several other concepts in probability and measure theory can be expressed in terms of Markov categories, including almost sure equality, convex mixtures, and conditional expectations.

Dynamical systems, invariant and ergodic states. Let G be a monoid or a group. A *dynamical system* in a Markov category C indexed by G is a functor from the one-object category defined by G to C. Explicitly, it consists of an object X of C and, for each element $g \in G$, an assigned morphism $X \to X$, which we also denote by g, such that the identity element of G is assigned the identity morphism 1_X and multiplication in G becomes composition in C. As special cases,

- if $\mathsf{C} = \mathsf{Meas}$, we speak of a *measurable* dynamical system;
- if $\mathsf{C} = \mathsf{Stoch}$, we speak of a *stochastic* dynamical system.

Often $G = \mathbb{N}$ and so the action is generated by a single morphism $X \to X$.

A state $p : I \to X$ is called *invariant* if, for every $g \in G$, $g \circ p = p$. In Stoch, the invariant states of a dynamical system are exactly its invariant measures. A *Kolmogorov colimit* X_{inv} of the action of G on X is a colimit of the diagram whose universal property respects determinism, i.e. which restricts to a colimit in the subcategory of deterministic morphisms. It can be proved that the space X equipped with the σ-algebra of *invariant sets* is a Kolmogorov colimit in Stoch whenever the action is deterministic. An invariant state $p : I \to X$ is called *ergodic* if the resulting state on X_{inv} is deterministic. In Stoch this means that the measure is invariant and zero-one-valued on every invariant set, this coincides with the usual definition of ergodic measure, so that the traditional definition can be given a clean categorical interpretation.

Main statement. Here is the synthetic version of the ergodic decomposition theorem.

Theorem 1. *Let C be a Markov category. Let X be a regular object, let G be a monoid acting on X deterministically, and suppose that the Kolmogorov colimit of the action exists. Then every invariant state of X can be written as a composition $f \circ m$ such that f is m-almost surely ergodic.*

We can instantiate this result for $\mathsf{C} = \mathsf{Stoch}$, and take as regular object X a standard Borel space. The Kolmogorov colimit of an action exists and is given by the invariant σ-algebra. The statement that we obtain is the traditional ergodic decomposition theorem (compare e.g. with [4, Proposition 4]):

Corollary 2. *Let G be a group or monoid acting deterministically on a standard Borel space X. Then every G-invariant measure of X can be written as a mixture of ergodic ones.*

References

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