Typed and Stratified Models with Slice Categories

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ABSTRACT

Categories of Petri nets are well-known tools in the applied category theory toolbox. In this technical demonstration, we present an application of these category-theoretic ideas to typed and stratified epidemiological modeling. Here a particular Petri net is selected to be the *type system*, and we define a *typed Petri net* to be an object in the category of Petri nets sliced over the type system. A product in this category is called a *stratified Petri net*. We highlight how these mathematical ideas reproduce results in the epidemiological literature and provide scientists with a software system that can use scientific knowledge to partially automate complex model construction. Finally, we demonstrate the practicality of these abstractions by implementing them in the Julia software package AlgebraicPetri.jl. This demonstration applies results previously published and available at https://github.com/AlgebraicJulia/Structured-Epidemic-Modeling/.

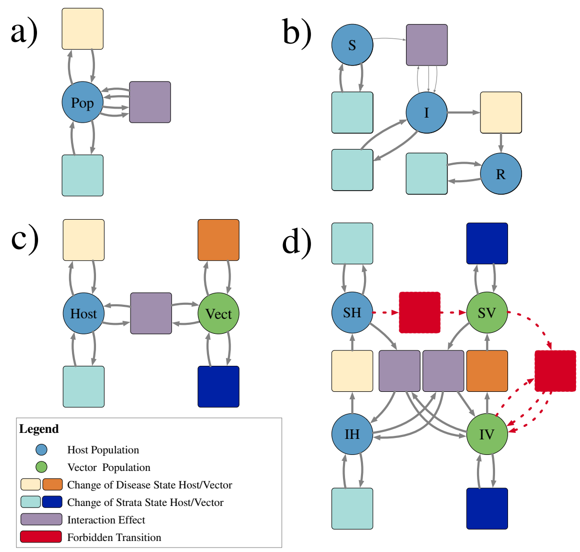


Figure 1: (a) A type system, $P_{\text{infectious}}$, representing Petri nets for infectious diseases. It has a single place type and three transition types corresponding to changes in infection status (yellow), changes between strata (blue), and interactions (purple). (b) An SIR disease model typed by $P_{\text{infectious}}$. Colors indicate the type of each place and each transition. (c) A type system, $P_{\text{vector-borne}}$, representing Petri nets for vector-borne diseases. It has two place types corresponding to the vector and host populations and specifies that interactions can only occur between vectors and hosts. (d) An SIS disease model typed by $P_{\text{vector-borne}}$. The transitions indicated in red have no valid typing and thus are forbidden by the type system.

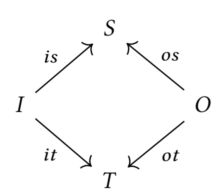
1 MOTIVATION

A recurring theme in scientific modeling is the importance of stratified models, in which local dynamics are reproduced in multiple strata and strata interact according to a specified scheme. For example, [2] compares stratified models defined by a choice of local epidemiological dynamics (SIR, SIS, and Ross-Macdonald) and a choice of models of movement between subpopulation (flux and simple trip). In the by-hand approach taken in [2], the adjustments to the disease models are done manually and do not express a formal relationship between the adjusted models and their component disease and movement models. By contrast, our approach to model stratification formalizes this construction and typed Petri nets offer a general methodology for stratifying models. Furthermore, our software implementation of the mathematical ideas presented in this section can then be applied to automatically generate the stratified models from the palette of component models, and thereby greatly reduces the size of the Petri nets that must be encoded by hand. Additionally, this approach makes it easier to extend the methods of [2], since new stratification schemes, once defined and typed, can be seamlessly integrated into the model construction, calibration, and analysis pipeline, instead of requiring experts to manually adjust each candidate disease model by each candidate movement model. These results were previously published in [4].

2 TYPED PETRI NETS

Let Petri be the category of whole-grain Petri nets [3, Section 2.2] defined by:

- Objects are diagrams of finite sets of the following shape:



- Morphisms are etale maps, in other words maps of places, transitions, input arcs, and output arcs that preserve the arities of the arcs and respect the sources and targets of the arcs.

Given a Petri net P_{type}, we call an object of the slice category $\mathsf{Petri}/P_{\text{type}}$ a *typed Petri net* and P_{type} the *type system*. Figure 1 gives examples of type systems and typed Petri nets.

The mathematical features of slice categories guarantee important modeling features. First, typed Petri nets are practical for model checking. A Petri net typed by $P_{\text{infectious}}$ assigns each transition to be a spontaneous change in infection, a spontaneous change in strata, or an interaction. The type of a transition must be consistent with the number of input and output arcs connected to it. For example, a typing by $P_{\text{infectious}}$ ensures that a transition with interaction type has two inputs and two outputs. Second, typed Petri nets facilitate high-level critiques of a model. For example, a model typed by $P_{\text{vector-borne}}$ cannot incorporate vertical transmission from parents to offspring or sexual transmission in either hosts or vectors. This property may contradict known transmission pathways for a specific disease and thus motivate a revision of the model and the type system. Third, features of the type system may also directly translate into features of the typed Petri net. For example, because each transition in $P_{\text{infectious}}$ has the same number of inputs and outputs, any Petri net typed by $P_{\text{infectious}}$ conserves the total population over time. Finally, typed Petri nets provide guardrails for model composition. The category $\mathsf{Petri}/P_{\text{type}}$ can be used to develop a category of open typed Petri nets following the strategy for oridinary open Petri nets developed in [1]. In this category of open typed Petri nets, only species of the same type can be identified in a composite.

3 STRATIFIED MODELS

Mathematically, a stratified model is a *pullback* of whole-grain Petri nets, or equivalently a *product* in the slice category over the given type system. Figure 2(b) gives an example of stratifying the SIR model by a model of quarantine/isolation status. In the stratified model, the place (S, Q) represents the susceptible and quarantining population while $(S, \sim Q)$ represents the susceptible and not quarantining population. The transition mediating the places $(S, \sim Q)$ and $(I, \sim Q)$ represents the infection transition in the SIR model and the interaction between non-quarantining people in the stratification scheme. Because these transitions are both of interaction type and mediate species of the same type, they are paired in the stratified model.

Properties of these well-studied categorical formalisms immediately verify useful properties of stratified Petri nets. For example, consider stratifying a disease model by quarantine status and by spatial dynamics. Since pullback is an associative and commutative binary operation, the order of stratification does not affect the final model. That is, the following procedures are equivalent: stratifying the disease model by spatial dynamics and then by quarantine status, stratifying the disease model by quarantine status and then by spatial dynamics, and stratifying the disease model by the stratification of spatial dynamics by quarantine status.

Our software implementation of the mathematical ideas presented in this section can then be applied to automatically generate the stratified models from the palette of component models. Figure 3 depicts a stratified Petri net automatically produced using AlgebraicPetri. This Petri net is the SIR disease model stratified by the simple trip model of population dynamics, one of the primary examples from [2]. The Jupyter notebook at https://github.com/AlgebraicJulia/Structured-Epidemic-Modeling/blob/main/stratification.ipynb contains a palette of disease models and stratification schemes and demonstrates using the `pullback` method defined in Catlab.jl to produce stratified Petri nets. Furthermore, it reproduces the stratified models defined in [2].

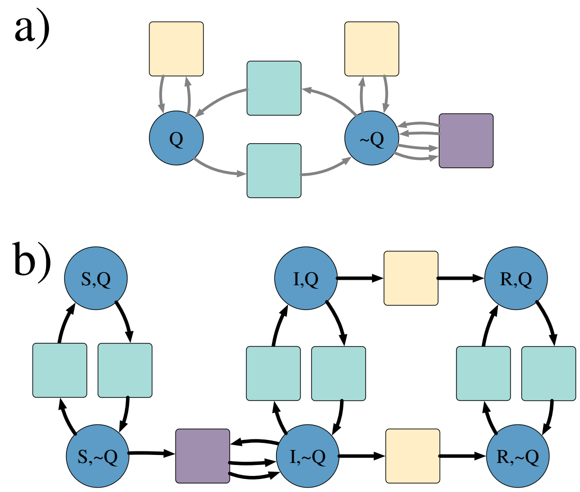


Figure 2: (a) A model of quarantine/isolation status. Interactions cannot occur between individuals who are quarantining/isolating. (b) The stratified model of the SIR model depicted in Figure 1(b) and a quarantining/isolation model depicted in (a).

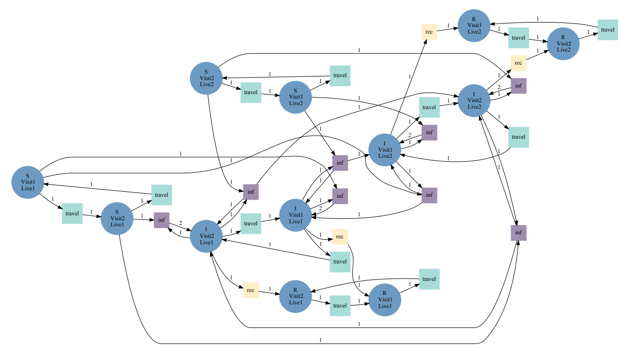


Figure 3: A stratified Petri net automatically produced by AlgebraicPetri. The colors of the transitions indicate their type.

4 CONCLUSION

Ultimately, a domain-specific typing can guarantee meaningful properties of a model and prevent novice users or automated systems from generating models that contradict common sense or domain expertise. Furthermore, the categorical abstraction of slice categories standardizes the definition of model stratification, and its implementation in AlgebraicPetri automates the construction of stratified models under the constraints of the expert-chosen type system. Because the size of stratified models grows quadratically with respect to the sizes of the component models, this framework streamlines the accurate implementation of stratified models as well as the clear communication and critique of stratified models via their components.

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