A comonadic account of Feferman-Vaught-Mostowski theorems (extended abstract)

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Abstract

Feferman-Vaught-Mostowski (FVM) composition theorems demonstrate when an operation on models preserves equivalence in a logic. In this presentation, we show that FVM theorems are an application of lifting the operation, on a category of models, to the Kleisli and Eilenberg-Moore category for a comonad. In particular, Spoiler-Duplicator game comonads, introduced by Abramsky, Dawar and Wang, and developed by Abramsky and Shah, internalize model comparison games in finite model theory and consequently provide semantics for logical equivalence. Our axiomatic account of FVM composition theorems when instantiated to a particular game comonad recover the usual FVM theorem for the logic associated to the Spoiler-Duplicator game comonad. Moreover, as these game comonads also provide semantics for the positive existential and counting quantifier variants of the logic, we also obtain new FVM theorems for these variants. As a byproduct of presenting the liftings of these operations succinctly, we employ the notion of bimorphism that not only applies to our scenario, but also generalizes bilinear maps in classic monad theory.

FVM theorems Feferman-Vaught-Mostowski (FVM) theorems characterize how logical equivalence behaves under composition and operations on models [Mos52, FV67]. The typical form of an FVM theorem states that, for a logic $\mathcal{L}$, and n-ary operation H on models of $\mathcal{L}$:

$$\text{if } A_i \equiv_{\mathcal{L}} B_i \text{ for all } 1 \leq i \leq n, \text{ then } H(A_1, \ldots, A_n) \equiv_{\mathcal{L}} H(B_1, \ldots, B_n) \tag{1}$$

where $\equiv_{\mathcal{L}}$ is the equivalence relation identifying all models that satisfy the same $\mathcal{L}$-sentences, i.e. $A \equiv_{\mathcal{L}} B \Leftrightarrow \forall \varphi \in \mathcal{L}, A \vDash \varphi \Leftrightarrow B \vDash \varphi$. For example, the first instances of FVM theorems showed that logical equivalence in first-order logic $\equiv_{\mathbf{FO}}$ behaves well with coproducts, $H = +$, [Mos52] and products, $H = \times$, [FV67] of models. Beyond applications in model theory [Gur85], FVM theorems are used in algorithmic meta-theorems [Cou90, CMR00, CE12, Mak04]. For example, Courcelle's theorem relies on FVM theorems that express how equivalence in monadic second-order logic behaves under operations that build graphs of bounded tree-width and bounded clique-width. In our forthcoming paper [JMS22], we provide an axiomatic framework for proving FVM theorems of the form (1). The core idea of the framework is to provide conditions for when an operation H lifts to the Kleisli and Eilenberg-Moore categories for a comonad on the category of models. The generality of these conditions then allow us to apply these axioms to one of the many game comonads recently discovered in [AS21, ADW17, AM21, CD21].

Game comonads Model comparison games, such as the Ehrenfeucht-Fraïssé game [Ehr61, Fra55], pebbling [KV92], and bisimulation games [HM80], are a key tool in finite model theory used to show inexpressiblity of properties in a logic. These resource indexed games are played on models and are defined such that one player, Duplicator, has a winning strategy in a game whenever the models are equivalent in an associated logic limited by some syntactic resource such as quantifier rank, variable count, or modal depth. Recent work [ADW17, AS21] has introduced a way of internalizing these games as comonads over the category of models. For example, the Ehrenfeucht-Fraïssé comonad $(\mathbb{E}_k, \varepsilon, \delta)$ [AS21] is defined on a category of σ-structures $\mathcal{R}(\sigma)$, where σ is a relational signature. The elements of $\mathbb{E}_k A$, for a σ-structure A, represents moves in the one-sided variant of the k-round Ehrenfeucht-Fraïssé game, namely $\mathbb{E}_k A$ consists of non-empty lists of elements in A of length $\leq k$. Morphisms in the Kleisli and Eilenberg-Moore categories of $\mathbb{E}_k$ correspond to Duplicator's winning strategies in the k-round two-sided, one-sided, and bijection variants of the Ehrenfeucht-Fraïssé game. Duplicator winning strategies in these games correspond to equivalence in first-order logic up to quantifier rank $\leq k$, $\mathbf{FO}_k$, the positive existential fragment $\exists^+\mathbf{FO}_k$ of $\mathbf{FO}_k$, and $\mathbf{FO}_k$ extended with counting quantifiers $\#\mathbf{FO}_k$. Thus, by the results in [AS21] equivalence in these logics without equality[1] are characterized in terms of $\mathbb{E}_k$ such that for all structures $A, B \in \mathcal{R}(\sigma)$:

(C1) $A \leftrightarrows_{\mathbb{E}_k} B \Leftrightarrow A \equiv_{\exists^+\mathbf{FO}_k} B$

(C2) $A \leftrightarrow_{\mathbb{E}_k} B \Leftrightarrow A \equiv_{\mathbf{FO}_k} B$

(C3) $A \cong_{\mathbf{Kl}(\mathbb{E}_k)} B \Leftrightarrow A \equiv_{\#\mathbf{FO}_k} B$ (with A, B assumed to be finite)

where $\leftrightarrows_{\mathbb{E}_k}$ denotes the existence of Kleisli morphisms $\mathbb{E}_k A \to B$ and $\mathbb{E}_k B \to A$, and $\cong_{\mathbf{Kl}(\mathbb{E}_k)}$ denotes isomorphism in the Kleisli category $\mathbf{Kl}(\mathbb{E}_k)$ of $\mathbb{E}_k$. Evidently from (C1) and (C3), one way to obtain FVM theorems for $\exists^+\mathbf{FO}_k$ and $\#\mathbf{FO}_k$ with operation $H\colon \mathcal{R}(\sigma) \to \mathcal{R}(\sigma)$ of the form in equation (1) is to construct a lifting $\hat{H}\colon \mathbf{Kl}(\mathbb{E}_k) \to \mathbf{Kl}(\mathbb{E}_k)$ of H to $\mathbf{Kl}(\mathbb{E}_k)$. Namely, we require that $\hat{H}F_{\mathbb{E}_k} \cong F_{\mathbb{E}_k}H$ where $F_{\mathbb{E}_k}\colon \mathcal{R}(\sigma) \to \mathbf{Kl}(\mathbb{E}_k)$ is the identity on objects and takes a morphism $f\colon A \to B \in \mathcal{R}(\sigma)$ to Kleisli morphism $f \circ \varepsilon\colon \mathbb{E}_k A \to B$. By standard constructions in category theory (see e.g. [Jac94, MM07]), such a lifting $\hat{H}$ is equivalent to the existence of a Kleisli law $\kappa\colon \mathbb{E}_k H \to H\mathbb{E}_k$. To recover the FVM theorem for $\mathbf{FO}_k$ with operation H, additional axioms are needed. This is because the relation $\leftrightarrow_{\mathbb{E}_k}$ in (C2) is defined as the existence of a notion of open map bisimulation for the Eilenberg-Moore category $\mathbf{EM}(\mathbb{E}_k)$, detailed in [AR21], between the cofree $\mathbb{E}_k$-coalgebras $F^{\mathbb{E}_k}(A)$ and $F^{\mathbb{E}_k}(B)$. In particular, the Kleisli law κ must satisfy an additional equalizer requirement in order to define a lifting $\bar{H}\colon \mathbf{EM}(\mathbb{E}_k) \to \mathbf{EM}(\mathbb{E}_k)$ of H to $\mathbf{EM}(\mathbb{E}_k)$. Furthermore, another couple of axioms are necessary to ensure this lifting $\bar{H}$ behaves well with respect to the bisimulation relation $\leftrightarrow_{\mathbb{E}_k}$.

The axiomatic framework for FVM theorems that we develop, generalizing this story about $\mathbb{E}_k$, applies to a wide range of operations H and comonads over $\mathcal{R}(\sigma)$. Namely, the framework is flexible enough to work with n-ary operations H which take input structures and output structures across different signatures σ. The class of comonads in the scope of our framework allows investigation of operations for any of the logics captured by other Spoiler-Duplicator comonads such as the pebbling comonad [ADW17], the modal comonad [AS21], comonads for guarded fragments [AM21], and comonads for logics with generalized quantifiers [CD21].

Bimorphisms and liftings Since a key part of this framework is liftings of operations to Kleisli and Eilenberg-Moore categories, the axioms are stated in terms Kleisli laws κ of these operations over comonads. For the operations we consider, the liftings induced by the Kleisli law are succinctly stated in terms of a notion of bimorphism. Bimorphisms not only apply to our scenario, but also generalize bilinear maps in classical monad theory as discussed in [Jac94, Sea13, MM07] by subsuming algebra morphisms.

[1] FVM theorems for $\mathbf{FO}_k$ with equality can also be handled in our framework by using relative liftings of $\mathbb{E}_k$ over a functor that makes equality an explicit part of our structures.

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