Categorical Semantics for Feynman Diagrams

Razin A. Shaikh[*]
Cambridge Quantum / Quantinuum
razin.shaikh@cambridgequantum.com

Stefano Gogioso[*]
Hashberg Ltd
quantum@hashberg.io

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Summary

We introduce a novel compositional description of Feynman diagrams, with well-defined categorical semantics as morphisms in a dagger-compact category. Our chosen setting is suitable for infinite-dimensional diagrammatic reasoning, generalising the ZX calculus and other algebraic gadgets familiar to the categorical quantum theory community.

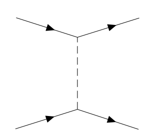
The Feynman diagrams we define look very similar to their traditional counterparts, but are more general: instead of depicting scattering amplitude, they embody the linear maps from which the amplitudes themselves are computed, for any given initial and final particle states. This shift in perspective reflects into a formal transition from the syntactic, graph-theoretic compositionality of traditional Feynman diagrams to a semantic, categorical-diagrammatic compositionality.

Because we work in a concrete categorical setting—powered by non-standard analysis—we are able to take direct advantage of complex additive structure in our description. This makes it possible to derive a particularly compelling characterisation for the sequential composition of categorical Feynman diagrams, which automatically results in the superposition of all possible graph-theoretic combinations of the individual diagrams themselves.

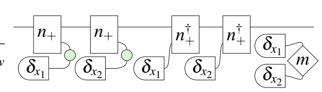
Categorical Feynman Diagrams

In this section, we develop a compositional, categorical description of Feynman diagrams as processes in *fHilb. The starting point for this description is the connection between Feynman diagrams and terms in Wick's expansion for the time-ordered products of Dyson's Formula. To concretely exemplify our construction, we work in scalar Yukawa theory: a complex scalar field of nucleons (n_+) and anti-nucleons (n_-), with "strong force" interactions mediated by a real scalar field of mesons (m).

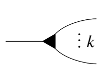
Consider the 2nd order Feynman diagram for nucleon-nucleon scattering shown on the right. This diagram corresponds to a term in Wick's expansion shown below. We translate this expression into a categorical diagram, obtaining a linearised version of the Feynman diagram as a process in *fHilb.



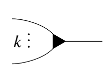
$$\frac{(-ig)^2}{2}\sum_{x_1,x_2}\frac{1}{\omega_{uv}^8}n_+^\dagger(x_1)n_+^\dagger(x_2)n_+(x_1)n_+(x_2)\overbrace{m(x_1)m(x_2)} \quad \rightsquigarrow \quad \frac{(-ig)^2}{2}\sum_{x_1,x_2}\frac{1}{\omega_{uv}^8}$$



This might not look like a Feynman diagram—and certainly possesses none of their intuitive beauty—but it describes the correct process. In order to understand the nature of the 2-dimensionality of Feynman diagrams, we introduce split and merge maps:



$: \mathscr{H}^{(\tau)} \to \mathscr{H}^{(\tau)} \otimes \ldots \otimes \mathscr{H}^{(\tau)}$



$: \mathscr{H}^{(\tau)} \otimes \ldots \otimes \mathscr{H}^{(\tau)} \to \mathscr{H}^{(\tau)}$

[*]Both authors contributed equally to this research.

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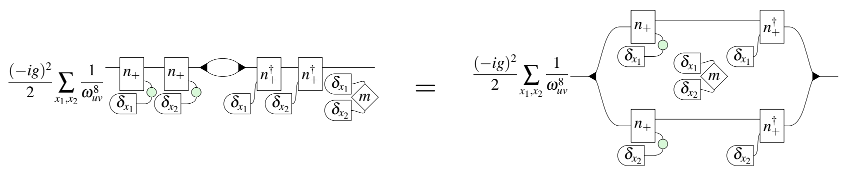
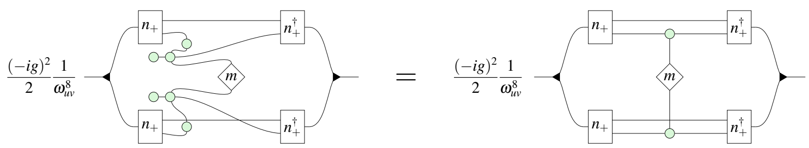
The relationship between these maps and ladder operators is captured by the sliding rules:

The split and merge maps for the same k are adjoints, with split maps being isometries. We introduce a split-merge pair in our categorical diagram and use the sliding rules:



Then we replace summations with spiders and simplify to obtain a categorical diagram that is functionally equivalent to the original Feynman diagram.



Composing Feynman Diagrams

As a calculation tool, traditional Feynman diagrams are inherently non-compositional. The diagrams themselves can be composed as graphs, by gluing, but the rules that turn them into amplitudes do not respect this compositional structure. Categorical Feynman diagrams, on the other hand, are prima-facie compositional: they are morphisms in a symmetric monoidal category, $^\star\mathbb{C}$-linear endomorphism of a certain hyperfinite-dimensional Hilbert space. Each categorical diagram directly embodies the full process that turns initial and final states into amplitudes. In its simplicity, however, sequential composition of categorical Feynman diagrams turns out to be surprising: because of sliding and commutation rules, composition automatically generates all possible ways of connecting intermediate legs in the 2-dimensional diagrams. We exemplify this phenomenon by composing two 2nd order nucleon scattering diagrams.