A Probability Monad on Measure Spaces (Extended Abstract)

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Abstract

We define a monad T on a category of measure spaces such that morphisms from 1 to T(X) correspond to probability density functions on X. The Kleisli category of this monad is dual to the category of commutative W*-algebras with normal positive unital maps as morphisms. This is an extension of the probabilistic Gelfand duality of Bart Jacobs and the author to W*-algebras.

The proof proceeds by showing that the category of W*-algebras is monadic over unital C*-algebras and also over Set, a result of interest in its own right. We then transfer the Radon monad, considered as a comonad on commutative C*-algebras, up to a comonad on commutative W*-algebras (this time with normal unital *-homomorphisms as morphisms), and obtain a monad on compact complete strictly localizable measure spaces by duality.

This is an extended abstract of the preprint [1].

Gelfand duality is the equivalence between the category **CHaus** of compact Hausdorff spaces and $\mathbf{CC^*Alg}^{\mathrm{op}}$, the opposite of the category of commutative unital C*-algebras, with unital *-homomorphisms as morphisms. One direction is given by the functor $C : \mathbf{CHaus} \to \mathbf{CC^*Alg}^{\mathrm{op}}$ that maps a space X to the algebra of complex-valued continuous functions, made into a C*-algebra with pointwise addition and multiplication of functions.

In [2], Bart Jacobs and the author described how to start with the Radon monad $\mathcal{R}$, the natural probability monad on the category **CHaus** of compact Hausdorff spaces, and define a variant of the Gelfand duality functor to give an equivalence between $\mathcal{K}\ell(\mathcal{R})$ and $\mathbf{CC^*Alg}^{\mathrm{op}}_{\mathrm{PU}}$, the category of commutative C*-algebras with positive unital maps (not required to preserve multiplication).

The category of commutative W*-algebras $\mathbf{CW^*Alg}$ is to measure spaces what $\mathbf{CC^*Alg}$ is to compact Hausdorff spaces, with the functor L^∞ playing the role of C. To be specific, we take the category **Meas** to have compact[1] complete strictly localizable measure spaces as objects and equivalence classes of nullset-reflecting measurable maps as morphisms, where the notion of equivalence[2] for measurable maps $f, g : (X, \Sigma_X, \nu_X) \to (Y, \Sigma_Y, \nu_Y)$ is that for all $T \in \Sigma_Y$ we have $\nu_X(f^{-1}(T) \triangle g^{-1}(T)) = 0$. Then $L^\infty : \mathbf{Meas} \to \mathbf{CW^*Alg}^{\mathrm{op}}$ is an equivalence, see for instance [4].

[1] This is a measure-theoretic notion, not the topological one, see [3, 342A (c)].

[2] In general this relation is coarser than equality almost everywhere, which would not make L^∞ a faithful functor.

A natural question arises as to whether there exists a monad T on **Meas** to play the role of the Radon monad and provide an equivalence $\mathcal{K}\ell(T) \simeq \mathbf{CW^*Alg}_{\mathrm{PU}}$. Since conditional expectations are morphisms in $\mathbf{CW^*Alg}_{\mathrm{PU}}$ this would give a monadic description of conditional expectations.

It turns out that this is the case, although the way of getting it is different. We start out with the observation that the proof of probabilistic Gelfand duality can be viewed under duality as showing that the inclusion $\mathbf{CC^*Alg} \hookrightarrow \mathbf{CC^*Alg}_{\mathrm{PU}}$ has a left adjoint given by $C\mathcal{S}$, the continuous functions on the state space, and that the coKleisli comparison functor for the comonad also written $C\mathcal{S}$ on $\mathbf{CC^*Alg}$ is an equivalence with $\mathbf{CC^*Alg}_{\mathrm{PU}}$.

The forgetful functors $\mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$ and $\mathbf{CW^*Alg}_{\mathrm{PU}} \to \mathbf{CC^*Alg}_{\mathrm{PU}}$ both have left adjoints, which are essentially the same, given by the *double dual* or *enveloping W*-algebra* $A \mapsto A^{**}$. For a W*-algebra of the form A^{**}, we know what the left adjoint of the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$ should be:

$$\begin{aligned}\mathbf{CW^*Alg}_{\mathrm{PU}}(A^{**}, B) &\cong \mathbf{CC^*Alg}_{\mathrm{PU}}(A, B) \cong \mathbf{CC^*Alg}(C(\mathcal{S}(A)), B) \\ &\cong \mathbf{CW^*Alg}(C(\mathcal{S}(A))^{**}, B).\end{aligned}$$

The trouble is that not every commutative W*-algebra is a double dual. However, the forgetful functor $\mathbf{CW^*Alg} \to \mathbf{CC^*Alg}$ and its left adjoint -^{**} form a monadic adjunction, and so every commutative W*-algebra is canonically a coequalizer of double duals. This allows us to produce a left adjoint to the inclusion $\mathbf{CW^*Alg} \hookrightarrow \mathbf{CW^*Alg}_{\mathrm{PU}}$. The coKleisli comparison functor for the comonad is an equivalence, essentially by the argument given in [5, Theorem 9] (in dual form). We then use the equivalence between $\mathbf{CW^*Alg}^{\mathrm{op}}$ and **Meas** to turn this into a monad T on **Meas** whose Kleisli category is equivalent to ${\mathbf{CW^*Alg}_{\mathrm{PU}}}^{\mathrm{op}}$.

As an explicit calculation, we are able to show that for all finite sets X, made into measure spaces with the counting measure, we have $L^\infty(T(X)) \cong C(2^\omega)^{**}$, and we find a compact complete strictly localizable measure space Y such that $L^\infty(Y) \cong C(2^\omega)^{**}$. If $\nu_d : \mathcal{P}(2^\omega) \to [0, \infty]$ is the counting measure and $\nu_c : \widehat{\mathcal{B}o(2^\omega)} \to [0, 1]$ is the completion of the usual probability measure describing an infinite sequence of independent fair coin flips, then

$$Y = (2^\omega, \mathcal{P}(2^\omega), \nu_d) + (2^\omega \times 2^\omega, \mathcal{P}(2^\omega) \otimes \widehat{\mathcal{B}o(2^\omega)}, \nu_d \otimes \nu_c),$$

where $\widehat{\mathcal{B}o(2^\omega)}$ is the completion of the Borel σ-algebra (with respect to ν_c) and $\otimes$ is Fremlin's c.l.d. product [6, Definition 251F].

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References

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