Diagrammatic presentations of enriched monads and theories for a subcategory of arities

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Building on Moggi's insight that monads model computational effects [1], the program of *algebraic computational effects* of Plotkin and Power begins with the idea that computational effects are *"realised by families of operations, with a monad being generated by their equational theory"* [2]. As such monads are usually enriched over a closed category $\mathscr{V}$, this program requires a robust theory of *presentations* of enriched monads by operations and equations. Work of Kelly, Power, and Lack [3, 4] provides a framework for presentations of enriched α-ary monads on a locally presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally presentable closed category $\mathscr{V}$, where the *arities* of the operations are α-presentable objects of $\mathscr{C}$ for a regular cardinal α. Recent generalizations involve working with a given *subcategory of arities* $\mathscr{J}$ in a $\mathscr{V}$-category $\mathscr{C}$ and considering enriched monads, theories, and pretheories defined relative to $\mathscr{J}$ [5, 6, 7]. In particular, Bourke and Garner [6] employ small subcategories of arities in locally presentable $\mathscr{V}$-categories in the case where $\mathscr{V}$ is locally presentable, but in this case the arities are still α-presentable for some α. The Kelly-Power-Lack approach to presentations has recently been generalized by Parker and the speaker [7] to apply to small *eleutheric* subcategories of arities in *locally bounded* $\mathscr{V}$-categories [8] over a locally bounded $\mathscr{V}$, thus removing the assumption of local presentability and so admitting a host of new examples in closed categories of relevance in computer science, topology, and analysis. Neither of the frameworks in [6] and [7] subsumes the other, and one may argue that none of the above frameworks entirely achieves the practical objective of presenting enriched monads directly in terms of individual operations, instead requiring the user to construct a signature internal to $\mathscr{C}$ or a pretheory enriched in $\mathscr{V}$.

In this talk, we establish a common extension of the above frameworks for presentations of enriched monads, and on this basis we introduce a flexible formalism for directly describing enriched algebraic structure borne by an object of a $\mathscr{V}$-category $\mathscr{C}$ in terms of what we call *parametrized $\mathscr{J}$-ary operations* and *diagrammatic equations*, for a suitable subcategory of arities $\mathscr{J}$. We introduce the notion of *diagrammatic $\mathscr{J}$-presentation*, and we show that each such presentation presents a $\mathscr{J}$-ary (or $\mathscr{J}$-nervous) $\mathscr{V}$-monad whose algebras may be described equivalently as objects of $\mathscr{C}$ equipped with specified parametrized operations, satisfying specified diagrammatic equations. By definition, a *$\mathscr{J}$-ary variety* is a $\mathscr{V}$-category of algebras for a diagrammatic $\mathscr{J}$-presentation, and we show that the category of $\mathscr{J}$-ary varieties is dually equivalent to the category of $\mathscr{J}$-ary $\mathscr{V}$-monads on $\mathscr{C}$.

We work in an axiomatic setting based primarily on the assumption that free algebras for $\mathscr{J}$-pretheories exist, and we establish a result to the effect that our axioms are in fact

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equivalent to the requirement that the given subcategory of arities *supports presentations* in an axiomatic sense. We show that our results on presentations of enriched monads are applicable in a wide variety of contexts in which $\mathscr{V}$ need not be locally presentable, such as in locally bounded closed categories $\mathscr{V}$ and various categories $\mathscr{C}$ enriched over such $\mathscr{V}$. In particular, among locally bounded closed categories one finds various convenient categories of relevance in programming language semantics, topology, analysis, and differential geometry, including all concrete quasitoposes [9] and various cartesian closed categories of topological spaces, such as compactly generated spaces.

We discuss examples of diagrammatic $\mathscr{J}$-presentations that illustrate their applicability for computational effects, including the global state algebras of Plotkin and Power [2] as well as various parametrized syntactic theories introduced by Staton for reasoning about algebraic effects [10]. We also discuss examples of diagrammatic $\mathscr{J}$-presentations in category theory, including presentations for internal categories and monoidal internal categories. Lastly, we define the *tensor product* of diagrammatic $\mathscr{J}$-presentations, which is relevant for combining algebraic computational effects (cf. [11]).

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