Compositionality as we see it, everywhere around us (abstract)

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Abstract

This is an **extended abstract** associated with the paper with the same title that will appear in a Festschrift honouring Andrei Khrennikov, edited by Arkady Plotnitsky and Emmanuel Haven.

There are different meanings of the term "compositionality" within science: what one researcher would call compositional, is not at all compositional for another researcher. The most established conception is usually attributed to Frege, and is characterised by a bottom-up flow of meanings: the meaning of the whole can be derived from the meanings of the parts, and how these parts are structured together.

Inspired by work on compositionality in quantum theory, and categorical quantum mechanics in particular, we propose the notions of Schrödinger, Whitehead, and complete compositionality. Accounting for recent important developments in quantum technology and artificial intelligence, these do not have the bottom-up meaning flow as part of their definitions.

Schrödinger compositionality accommodates quantum theory, and also meaning-as-context. Complete compositionality further strengthens Schrödinger compositionality in order to single out theories like ZX-calculus, that are complete with regard to the intended model. All together, our new notions aim to capture the fact that compositionality is at its best when it is 'real', 'non-trivial', and even more when it also is 'complete'.

At this point we only put forward the intuitive and/or restricted formal definitions, and leave a fully comprehensive definition to future collaborative work.

The area mostly associated with a formal definition of compositionality is formal linguistics, stated as follows:

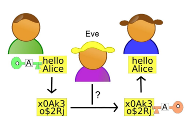
- The meaning of a whole (cf. sentence) should only depend on the meanings of its parts (cf. word meanings) and how they are fitted together (cf. grammar).

A prominent argument for this notion, put forward by Frege [10], is that without compositionality it would be impossible for us to understand a sentence that we have never heard before. Broadly speaking, this is of course indeed the case. However, interestingly Frege himself mentions in 'Grundlagen der Arithmetik' [9, 14]:

- "Never ask for the meaning of a word in isolation, but only in the context of a sentence."

This is called 'the context principle' [20], and indeed:

- In language, meanings are to a great extent induced by the context. For example, ambiguous words such as `Alice` could point at either of these two incarnations of `Alice`:



  We may disambiguate `Alice` by other words in the sentence or larger text, or by non-linguistic features, like the conversation taking place between two children impersonating fairytale characters, or between security protocol experts. Even within the restricted context of gothic rock fans `Alice` could still have two meanings, two cases being:

There are some more refined formulations of linguistic compositionality that aim to account for context-dependent word ambiguities, however, the following is harder to get around:

- In language, separation between word meanings and how they grammatically fit together doesn't always make sense, as the grammar can depend on the overall meaning, which may need additional context to be disambiguated. For example, in `black metal fan` the grammar depends on the meanings:[1]

[1] We took this particular example from [8], but there are of course many more.

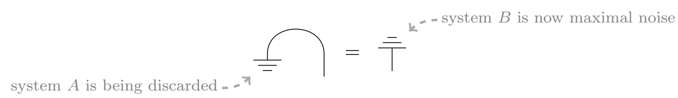
Using the language of [5], the respective grammatical parsings proofs are:

Hence there is no fixed grammar until the overall meaning has been unambiguously established.

In the linguistic context the fundamental problem with the above stated notion of compositionality is that it puts a special focus on bottom-up meaning-flows, the meaning of the whole being induced by the meanings of the parts. However, meaning-flows within a composite can equally well travel top-down along the same structure, that is, the meaning of the whole can contribute to meanings of parts. So what we are aiming for is some network that allows for such two-way meaning-flows, or even better put, a network structure that allows for meaning-relationships along the different scales.

In quantum theory the situation gets even more extreme:

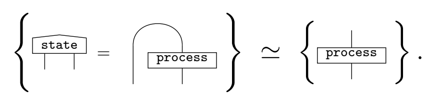
- For entangled states the idea that subsystems have meanings breaks down, and so does Frege compositionality: the pure Bell-state's sub-systems are in the state of total noise when conceived individually. In order to know that the overall state is the Bell-state, one needs the entire description of the Bell-state itself:



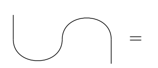
So there is no meaningful decomposition of the whole in terms of parts plus structure. Yet both language and quantum theory are perfectly compositional as far as we are concerned. Schrödinger singled out the nature of composition of systems in quantum theory not as just a, but as the defining feature of the theory [17]:

"Entanglement is not one but rather the characteristic trait of quantum mechanics."

We refer to a situation where the composite of two systems cannot be meaningfully described in terms of its parts as *non-trivial composition*. The Bell-states and any other maximally entangled states constitute an extremal example of non-trivial composition [6, 11], the possible states of two systems being isomorphic to the processes that each system can undergo:



where the isomorphism is established by the following equation:



What makes these kinds of states interesting is not what they are, but what one can do with them. This means that a general notion of compositionality should apply to general theories of processes, not just theories of states. The idea of focusing on processes instead of states goes back to the pre-Socatic Heraclitus' "panta rhei", a.k.a. everything flows, and in more recent times was advocated by Whitehead in "Process and Reality" [19]. His stance departs from our typical Western metaphysics in which we start with a static kinematic layer with dynamics as a secondary derivative [18]. Whitehead's work however did not explicitly consider a non-trivial *parallel* composition.

Definition 0.1. A *Schrödinger compositional theory* is a process theory [7] subject to two conditions:

- Composition is non-trivial, that is, the description of a whole cannot always be decomposed in any meaningful way.
- All processes have clear meaningful ontological counterparts in reality.

Note that neural nets do not obey this definition.

Definition 0.2. A *Whitehead compositional theory* is the same as a Schrödinger compositional theory except for the fact that composition is restricted to the sequential mode.

Definition 0.3. A *complete compositional theory* is a complete [1, 2, 12, 13, 15] Schrödinger compositional theory.

The ZX-, ZW- and ZH-calculi are examples of this. In the case of ZX-calculus by 'real' we mean that each spider corresponds to a physical process that may occur. We say 'may', as most spiders cannot be realised in a deterministic fashion: they can only occur as a branch in a non-deterministic process, e.g. an appropriately crafted measurement. In other words, they are not 'causal' [3, 4, 16].

Now, rather than considering all quantum processes, one could restrict to unitary ones. In that context, the ZX-spiders are not 'real' anymore and only occur as '1/2 of a CNOT-gate'. So now real processes are further decomposed in smaller bits. These smaller bits greatly facilitate computation as compared to the unitary processes.

Definition 0.4. *Lego compositionality* is a generalisation of Schrödinger compositionality in that processes may be broken down in smaller pieces that have no counterpart in reality.

Each of these definitions is by no means as general as they can be. In particular, our notion of process theory of [7] is still quite restrictive, not accounting for higher-order processes.

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