Making Modalities (Lax) Monoidal

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In 1958, Lambek introduced the logic of a residuated monoid[1], aka a monoidal biclosed category, to reason about syntax of natural language. This logic was among the first substructural logics, with no contraction or weakening. The lack of these properties was necessary to analyse fixed word order languages and the logic proved to be sufficient for analysing elementary sentential fragments. Later work of Moortgat, Morrill, and Jäger in the 90's, and Kanovitch et al. in the 21st century, showed that adding substructural modalities to Lambek's calculus increases its expressive power. The new logics were used to analyse non associativity, iterative conjunctives, parasitic gaps and various unbounded dependancies of natural language. In previous work, using Lambek calculus with soft subexponentials (**SLLM**) of [2], we extended the analysis to beyond sentential, to discourse phenomena such as anaphora and ellipsis [4]. We developed a categorical and vector space semantics [4, 3] in the style of the DisCoCat models of [1].

The categorical semantics of [4] was defined using the method of Melliès [5][2]. It is a monoidal biclosed category, $\mathscr{C}(\mathbf{SLLM})$, endowed with endofunctors M and P, each equipped with additional natural transformations corresponding ***multiplexing*** $(\pi_n : M \to \mathrm{id}^{\otimes n})_{n=1,2,\ldots,k_0}$, ***permutation*** $\sigma_{-_1,-_2} : P(-_1) \otimes \mathrm{Id}(-_2) \to \mathrm{Id}(-_2) \otimes P(-_1)$ and a *counit* $e : P \to \mathrm{Id}$. These endofunctors are the categorical semantics of the soft subexponential modalities ! and ∇ of **SLLM**, but due to their syntactic properties, they are neither oplax nor lax monoidal. Thus the natural preference of a category theorist that functors between monoidal categories be compatible with the monoidal products, fails. Furthermore this restricts the completeness of the models of **SLLM**, as the model of ! in [4] is in fact lax-monoidal. In this paper, we remedy these weaknesses by adding rules to the syntax of **SLLM** corresponding to lax-monoidality of M and P, and prove that in the result resulting calculus, dubbed **Monoidal SLLM**, the cut rule is eliminable.

We recall some notions from proof theory, suitably adapted to our purpose. A *sequent* is a pair (Γ, A), denoted $\Gamma \longrightarrow A$, where A is a single formula and Γ is a list of formulas. A *proof* or *derivation* of $\Gamma \longrightarrow A$ is a tree of sequents rooted in $\Gamma \longrightarrow A$, whose leaves are axioms $A \longrightarrow A$ and whose edges are rules of the calculus. For example, the following is a proof of $B, (B\backslash A)/C \longrightarrow A/C$ in Lambek calculus:

$$\dfrac{\dfrac{C \longrightarrow C \qquad \dfrac{B \longrightarrow B \quad A \longrightarrow A}{B, B\backslash A \longrightarrow A}\,\backslash_L}{B, (B\backslash A)/C, C \longrightarrow A}\,/_L}{B, (B\backslash A)/C \longrightarrow A/C}\,/_R \quad . \tag{1}$$

SLLM extends the language of Lambek calculus with two modalities $!, \nabla$ and the following rules:

[1] Technically, this was a residuated semigroup, and the following categorical constructions exist for residuated semigroups too. We adopt the monoid-approach in this paper as this is a more common structure, and suits the purposes of our construction.

[2] This is the construction of a categorical semantics of linear logic (**LL**), where the formulas of **LL** are objects, and the proofs of sequents in the logic are morphisms and cut is composition.

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$$\frac{\Gamma_1,\overbrace{A,\ldots,A}^{k\geq 1 \text{ times}},\Gamma_2 \longrightarrow C}{\Gamma_1, !A, \Gamma_2 \longrightarrow C}\ !_L \qquad \frac{A \longrightarrow C}{!A \longrightarrow !C}\ !_R \qquad \frac{\Gamma_1, A, \Gamma_2 \longleftrightarrow C}{\Gamma_1, \nabla A, \Gamma_2 \longleftrightarrow C}\ \nabla_L \qquad \frac{A \longrightarrow C}{\nabla A \longrightarrow \nabla C}\ \nabla_R$$

$$\frac{\Gamma_1, \nabla A, \Gamma_2, \Gamma_3 \longrightarrow C}{\Gamma_1, \Gamma_2, \nabla A, \Gamma_3 \longrightarrow C}\ \nabla_{E_1} \qquad \frac{\Gamma_1, \Gamma_2, \nabla A, \Gamma_3 \longrightarrow C}{\Gamma_1, \nabla A, \Gamma_2, \Gamma_3 \longrightarrow C}\ \nabla_{E_2}$$

One particular rule of interest when working in a sequent calculus is the *cut rule*

$$\frac{\Gamma \longrightarrow A \quad \Sigma[A] \longrightarrow B}{\Sigma[\Gamma] \longrightarrow B}\ cut \tag{2}$$

where the notation $\Sigma[A]$ means that one of the formulas in Σ is A, and that the *cut formula* A is replaced by Γ in the new structure $\Sigma[\Gamma]$. As the notation suggests, the cut rule corresponds to the *composition* in a category, in this case our syntactic category $\mathscr{C}(\mathbf{SLLM})$.

In terms of bottom-up proof search, *cut* corresponds to the introduction of an intermediate goal (or *lemma*) A. While cuts are an ubiquitous and useful tool for the working mathematician, they pose an almost unsurmountable problem for *algorithmic* proof search: Since the cut formula bears no obvious relation to the lower sequent $\Sigma[\Gamma]$, it has to be "guessed" by the search algorithm - and there are infinitely many options. However, by setting up a sequent calculus carefully one can often show that the cut rule is actually redundant, in the sense that every proof can be rewritten so that it does not use *cut* anymore. For **SLLM**, such a *cut elimination theorem* was shown in [2].

Returning to our categorical motivation, we make **SLLM** monoidal by adding the two rules below:

$$\frac{\Gamma_1, !(A\cdot B), \Gamma_2 \longrightarrow C}{\Gamma_1, !A, !B, \Gamma_2 \longrightarrow C}\ !_m \qquad \text{and} \qquad \frac{\Gamma_1, \nabla(A\cdot B), \Gamma_2 \longrightarrow C}{\Gamma_1, \nabla A, \nabla B, \Gamma_2 \longrightarrow C}\ \nabla_m \tag{3}$$

These correspond to lax monoidality of ! and ∇ modalities. We prove that, as in **SLLM**, cuts can be eliminated in the resulting system **Monoidal SLLM**. The proof proceeds in two steps. We first show that **Monoidal SLLM** is equivalent to a more general version of **SLLM** in which a standard cut-elimination technique is . We then show that cut-free proofs of the latter calculus can be simulated by cut-free proofs in **Monoidal SLLM** using the lax monoidal rules. We also show that adding the oplax rules (these are the rules in (3) but hold in the opposite direction) yields a calculus in which cuts *cannot* be eliminated.

Adding lax-monoidality to the syntax of the modalities of **SLLM** and maintaining cut-elimination is a novel result in the proof theory of noncommutative substructural logics and is inspired entirely by its categorical semantics. We believe the linguistic results stay intact, but leave a full study to future work.

References

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