Exchangeability and the Radon Monad: Probability Measures, Quantum States and Multisets

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1 Introduction

De Finetti theorems give equivalences between exchangeable sequences of random processes and sequences generated by drawing one experiment at random and then repeating it over and over. The main results of this paper are that both classical and quantum de Finetti constructions have universal properties as categorical limits in particular categories. Additionally, we show that exchangeability can be considered explicitly with products or implicitly using multisets. The categorical concept that bridges these cases is the Radon probability monad: its Kleisli category is a category of probability kernels and, from the perspective of categorical probability theory, a Markov category; whilst quantum state spaces live in its category of Eilenberg-Moore algebras. Our de Finetti diagrams and their limits are situated in these two categories.

The categorical structure shows how de Finetti's theorem allows us to pass from relatively simple structures, such as sequences of probabilities and multisets, to more complex or conceptual spaces, such as spaces of measures (Theorem 3.1, Theorem 3.2) and spaces of quantum states (Theorem 4.1). Thus de Finetti's philosophical arguments [1] take an explicit categorical form. In this way, our work begins a connection between axiomatic approaches to categorical probability (e.g. [2, 3]), to quantum quantum probability (e.g. [11, 14, 8]), and categorical explorations of states and effects (e.g. [4, 15]). For further details, with a quantum focus, see [12].

2 The Radon Monad

The Radon monad $\mathcal{R}$ is a probability monad on **CH**, the category of compact Hausdorff spaces. Like all probability monads, $\mathcal{R}$ takes an object $X \in \mathbf{CH}$ to an object, $\mathcal{R}(X)$, of measures on X. We are only interested in measures that are compatible with compact subspaces of X, so we take the set of Radon measures on the Borel σ-algebra of X. It is topologised such that it is again compact and Hausdroff (See [4] for details).

On morphisms, it acts by pushing forward measures by functions. The unit of the monad takes a point $x \in X$ to the delta distribution δ_x solely supported at x. The multiplication $\mathrm{mult}\colon \mathcal{R}(\mathcal{R}(X)) \to \mathcal{R}(X)$ is marginalisation: for $\Phi \in \mathcal{R}(\mathcal{R}(X))$ and $A \subset X$ Borel, $\mathrm{mult}(\Phi)(A) = \int_{\mu \in \mathcal{R}(X)} \mu(A)\,\mathrm{d}\Phi(\mu)$

We recall that the Kleisli category of the Radon monad is a semi-cartesian monoidal category, i.e. a good classical setting for categorical probability [2]. The Kleisli category can be thought of as the free algebras. The Eilenberg-Moore category of *all* algebras for the Radon monad faithfully includes the state spaces of C*-algebras, and hence quantum probability [4].

3 Two Classical de Finetti Theorems as Categorical Limits

Our first theorem shows that de Finetti characterizes $\mathcal{R}(X)$ as a limit of a diagram of projections and permutations in the Kleisli category.

Exchangeability describes finite-permutation invariance of sequences of random processes. Let X be a measurable space and let $\mu \in \mathcal{R}(X^n)$. For each permutation $\sigma \in \mathcal{S}_n$, there is a homeomorphism

$$\eta_\sigma\colon X^n \to X^n \qquad (x_1, x_2, \ldots, x_n) \mapsto (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}).$$

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μ is called *symmetric* if, for every permutation $\sigma \in \mathcal{S}_n$, we have that $(\eta_\sigma)_*\mu = \mu$. A measure $\mu \in \mathcal{R}\left(X^{\mathbb{N}}\right)$ is *exchangeable* if the measure it defines on X^n by projection is symmetric for all $n \in \mathbb{N}$.

A classical de Finetti theorem tell us that exchangeable sequences of measures are in correspondence with measures on independent and identically distributed sequences [1, 5].

Let $\mathbf{I}_{\text{inj}}$ be the category of the finite sets $\{1,\ldots,n\}$ for $n \in \mathbb{N}$ and injections between them.

Theorem 3.1. *In the Kleisli category of the Radon monad and for any $X \in \mathbf{CH}$, the diagram $\left(\mathbf{I}_{\text{inj}}\right)^{\text{op}} \to \mathcal{K}l\left(\mathcal{R}\right)$ which takes $\{1\ldots n\}$ to X^n and each injection $\tau\colon \{1\ldots n\} \to \{1\ldots m\}$ to the Kleisli map $\eta_\tau\colon X^m \rightsquigarrow X^n$ given by $\eta_\tau(x_1,\ldots,x_m) = \delta_{\left(x_{\tau(1)},x_{\tau(2)},\ldots,x_{\tau(n)}\right)}$ has limit $\mathcal{R}\left(X\right)$ with maps $\text{iid}_n\colon \mathcal{R}\left(X\right) \rightsquigarrow X^n$ given, for measurable $A \subset X^n$, by*

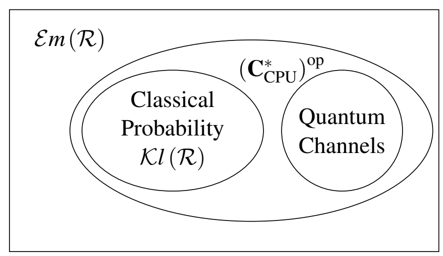
$$\text{iid}_n(\mu)(A) = (\underbrace{\mu \times \cdots \times \mu}_{n\ \text{times}})(A).$$

Our second theorem gives an alternative view on the characterization of $\mathcal{R}(X)$ in terms of the limit of a chain of spaces of multisets. The multisets $\mathcal{M}\left[n\right]\left(X\right)$ of order n with elements in X can be topologized as a coequalizer (equivalently in **Top** or **CH**) of the maps $X^\sigma\colon X^n \to X^n$ for all $\sigma \in \mathcal{S}_n$. The elements of $\mathcal{M}\left[n\right]\left(X\right)$ are formal sums $\sum_{x\in X} m(x)\left|x\right\rangle$ for $m\colon X \to \mathbb{N}$, $\sum_{x\in X} m(x) = n$ and measures on $\mathcal{M}\left[n\right]\left(X\right)$ correspond with symmetric measures on X^n. We consider the chain of continuous maps $\text{DD}_n\colon \mathcal{M}\left[n+1\right]\left(X\right) \to \mathcal{R}\left(\mathcal{M}\left[n\right]\left(X\right)\right)$ for $n \in \mathbb{N}$ which takes a multiset given by m to a multiset given by randomly dropping an element: $\sum_{x\in\text{supp}(m)} \frac{1}{n}\delta_{m-1_x}$, where $1_x(x) = 1$ and is 0 otherwise.

Theorem 3.2. *Let X be a compact Hausdorff space. The limit of the diagram $\omega \to \mathcal{K}l\left(\mathcal{R}\right)$ which takes n to $\mathcal{R}\left(\mathcal{M}\left[n\right]\left(X\right)\right)$ and $n+1 \to n$ to DD_n is $\mathcal{R}\left(X\right)$. Further, this limit is reflected by the forgetful functor $\mathcal{E}m\left(\mathcal{R}\right) \to \mathcal{K}l\left(\mathcal{R}\right)$.*

This is inspired by the central result of [10], which shows a similar limit for $X = 2$. Possible extensions to Markov categories are suggested by [9].

4 A Quantum de Finetti Theorem as a Categorical Limit

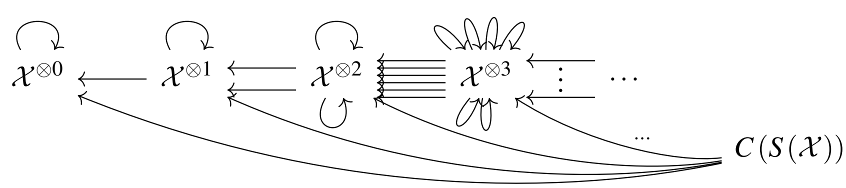


As shown in the Venn diagram, by moving to the algebras of the Radon monad $\mathcal{E}m\left(\mathcal{R}\right)$, we are able to include quantum and classical probability. An intermediate region is $\mathbf{C}^*$-algebras, their state spaces and completely positive maps, which are recognised as providing a model for hybrid quantum-classical systems. It is helpful to bear in mind all four categories: the uniting category of $\mathbf{C}^*$-algebras and completely positive unital maps $(\mathbf{C}^*_{\text{CPU}})^{\text{op}}$ has good monoidal structure, whereas other categorical structures in $\mathcal{E}m\left(\mathcal{R}\right)$ are more canonical for calculations and relate our results back to classical probability.x

The set of states, $S\left(\mathcal{X}\right) := \mathbf{C}^*_{\text{CPU}}(\mathcal{X},\mathbb{C})$, of a $\mathbf{C}^*$-algebra $\mathcal{X}$ can be given a topology such that it forms a convex compact Hausdorff space. Defining $S = \mathbf{C}^*_{\text{CPU}}(-,\mathbb{C})$ with this additional structure gives a functor from $(\mathbf{C}^*_{\text{CPU}})^{\text{op}}$ to $\mathcal{E}m\left(\mathcal{R}\right)$ (see [4] for details).

Our main result in [12] is that the quantum analogue of De Finetti's theorem [7, 6, 13] gives a universal property for the space of states.

Theorem 4.1. *Let $\mathcal{X}$ be a $\mathbf{C}^*$-algebra. We define a diagram $\left(\mathbf{I}_{\text{inj}}\right)^{\text{op}} \to (\mathbf{C}^*_{\text{CPU}})^{\text{op}}$ by taking $\{1,\ldots,n\}$ to $\mathcal{X}^{\otimes n}$ and an injection $\tau\colon \{1,\ldots,n\} \hookrightarrow \{1,\ldots,m\}$ to the map $\eta_\tau\colon \mathcal{X}^{\otimes n} \to \mathcal{X}^{\otimes m}$ given by taking $A_1 \otimes\cdots\otimes A_n$ to the element $B_1 \otimes\cdots\otimes B_m$ with $B_j = \begin{cases} A_i & \text{if } j = \tau(i) \\ 1 & \text{otherwise} \end{cases}$. The limit of this diagram in $(\mathbf{C}^*_{\text{CPU}})^{\text{op}}$ is $C\left(S\left(\mathcal{X}\right)\right)$.*



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