On Ramsey Theory, Category Theory and Entropy

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Close connections between various notions of entropy and the apparatus of category theory have been observed already in the 1980s and more vigorously developed in the past ten years. The starting point of this talk is the recent categorical understanding of structural Ramsey degrees, which then leads to a way to compute entropy of an object in a small category not as a measure of statistical, but as a measure of its combinatorial complexity. The new entropy function we propose, the Ramsey entropy, is a real-valued invariant of an object in an arbitrary small category. We require no additional categorical machinery to introduce and prove the properties of this entropy. Motivated by combinatorial phenomena (structural Ramsey degrees) we build the necessary infrastructure and prove the fundamental properties using only special partitions imposed on homsets.

1 Introduction

Ramsey theory is one of only a few mathematical theories whose reach goes from elementary problems that can be demonstrated to undergraduates, to a variety of surprisingly subtle applications in mathematical logic, set theory, finite combinatorics and topological dynamics. The successful generalization of Ramsey's theorem from sets (unstructured objects) to cardinals (special well-ordered chains) lead in the early 1970's to generalizing this setup to arbitrary first-order structures, giving birth to *structural Ramsey theory*. Interestingly, one of the first comprehensive texts published already in 1973 was Leeb's book [7] where structural Ramsey theory was presented using the language of category theory. Unfortunately, this point of view was relatively quickly pushed out of the focus of the research community and was replaced by the model-theoretic approach that is even today a dominant point of view.

Close connections between various notions of entropy and the apparatus of category theory have been observed already in the 1980's [6] and more vigorously developed in the past ten years [1, 2]. In particular, [8, Chapter 12] describes a general categorical construction which specialized to a real line produces the Shannon entropy.

In this talk we present some of the results from [11] and [12] and along the way demonstrate the way category theory can be employed to provide a deeper understanding of interrelationships between apparently unrelated notions such as Ramsey degrees and entropy of combinatorial structures. We propose a way to compute entropy of a combinatorial (finite first-order) structure not as a measure of statistical, but as a measure of its combinatorial complexity.

Two central features of every entropy function are continuity and the logarithmic property ($H(A,B) = H(A) + H(B)$). Due to the discrete nature of abstract categories (and, consequently, absence of any nontrivial intrinsic topology apart from the topology induced by the preorder $A \to B$ iff $\hom(A,B) \neq \varnothing$) in this paper we adopt the approach advocated in [9] where beside the additive property entropy is required to be monotonous. So, the entropy functions we are interested in are those with the logarithmic property ($H(A,B) = H(A) + H(B)$) which are monotonous ($A \to B \Rightarrow H(A) \leqslant H(B)$).

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2 Preliminaries

Structural Ramsey degrees are defined in terms of coloring "subobjects" using the fact that a subobject of B isomorphic to A can be identified by a set of "embeddings $A \to B$" that "differ by an automorphism of A". More precisely, let $\mathbf{C}$ be a locally small category whose morphisms are mono. For $A, B \in \mathrm{Ob}(\mathbf{C})$ define an equivalence relation $\sim_A$ on $\hom(A,B)$ as follows: for $f, g \in \hom(A,B)$ we let $f \sim_A g$ if there is an $\alpha \in \mathrm{Aut}(A)$ such that $f = g \cdot \alpha$. We then think of $\binom{B}{A} = \hom(A,B)/\sim_A$ as the *set of subobjects of B isomorphic to A*. (Note that if $f \in \hom(A,B)$ and $w \in \hom(B,C)$ then $w \cdot (f/\sim_A) = (w \cdot f)/\sim_A$; this justifies the correctness of the construction $w \cdot \binom{B}{A}$ that we shall use to define structural Ramsey degrees.) The requirement that morphisms in $\mathbf{C}$ be mono is not essential for the definition of these notions, but almost no interesting property of Ramsey degrees can be proved without this assumption.

For $A \in \mathrm{Ob}(\mathbf{C})$ let $\tilde{t}_{\mathbf{C}}(A)$, the *structural Ramsey degree of A in* $\mathbf{C}$, denote the least positive integer n, if such an integer exists, such that for all $k \in \mathbb{N}$ and all $B \in \mathrm{Ob}(\mathbf{C})$ there exists a $C \in \mathrm{Ob}(\mathbf{C})$ such that for every coloring $\chi : \binom{C}{A} \to k$ of subobjects of C isomorphic to A with k colors there is a $w \in \hom(B,C)$ such that $|\chi(w \cdot \binom{B}{A})| \leqslant n$. If no such integer exists, we write $\tilde{t}_{\mathbf{C}}(A) = \infty$. Analogously, for $A \in \mathrm{Ob}(\mathbf{C})$ let $t_{\mathbf{C}}(A)$, the *embedding Ramsey degree of A in* $\mathbf{C}$, denote the least positive integer n, if such an integer exists, such that for all $k \in \mathbb{N}$ and all $B \in \mathrm{Ob}(\mathbf{C})$ there exists a $C \in \mathrm{Ob}(\mathbf{C})$ such that for every coloring $\chi : \hom(A,C) \to k$ there is a $w \in \hom(B,C)$ such that $|\chi(w \cdot \hom(A,B))| \leqslant n$. If no such integer exists, we write $t_{\mathbf{C}}(A) = \infty$. An $A \in \mathrm{Ob}(\mathbf{C})$ is a *Ramsey object in* $\mathbf{C}$ if $\tilde{t}_{\mathbf{C}}(A) = 1$. A $B \in \mathrm{Ob}(\mathbf{C})$ is a *subramsey object in* $\mathbf{C}$ if $B \to A$ for some Ramsey object A. The following relationship between structural and embedding Ramsey degrees was proved for relational structures in [13] and generalized to this form in [10].

Proposition 2.1. *([13, 10]) Let $\mathbf{C}$ be a locally small category whose morphisms are mono and let $A \in \mathrm{Ob}(\mathbf{C})$. Then $t(A)$ is finite if and only if both $\tilde{t}(A)$ and $\mathrm{Aut}(A)$ are finite, and in that case $t(A) = |\mathrm{Aut}(A)| \cdot \tilde{t}(A)$.*

The following is a technical result from [12] which will form the basis for the key properties of the Ramsey entropy.

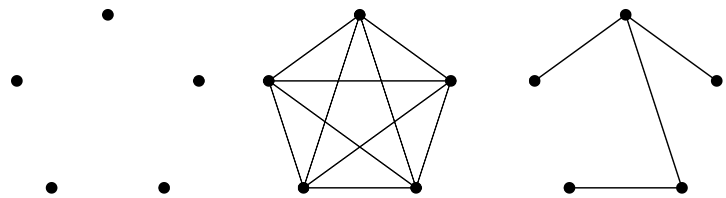
Theorem 2.2. *[12] Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories whose morphisms are mono and homsets are finite. Then the following holds for all $A_1 \in \mathrm{Ob}(\mathbf{C}_1)$ and $A_2 \in \mathrm{Ob}(\mathbf{C}_2)$:*

- $t_{\mathbf{C}_1 \times \mathbf{C}_2}(A_1, A_2) = t_{\mathbf{C}_1}(A_1) \cdot t_{\mathbf{C}_2}(A_2)$, *and*
- $\tilde{t}_{\mathbf{C}_1 \times \mathbf{C}_2}(A_1, A_2) = \tilde{t}_{\mathbf{C}_1}(A_1) \cdot \tilde{t}_{\mathbf{C}_2}(A_2)$.

Concerning the monotonicity of Ramsey degrees, the main result we shall need, Lemma 2.3, was first proved in [13] for amalgamation classes of structures, and reproved in [10] for locally small directed categories with amalgamation. We shall say that $\mathbf{C}$ is a category *with amalgamation* if for every span $B \xleftarrow{f} A \xrightarrow{g} C$ in $\mathbf{C}$ there is a cospan $B \xrightarrow{f'} D \xleftarrow{g'} C$ in $\mathbf{C}$ such that $f' \cdot f = g' \cdot g$.

Lemma 2.3. *(cf. [13, 10]) Let $\mathbf{C}$ be a category with amalgamation whose morphisms are mono and whose homsets are finite. Then for all $A_1, A_2 \in \mathrm{Ob}(\mathbf{C})$, if $A_1 \to A_2$ then $t(A_1) \leqslant t(A_2)$.*

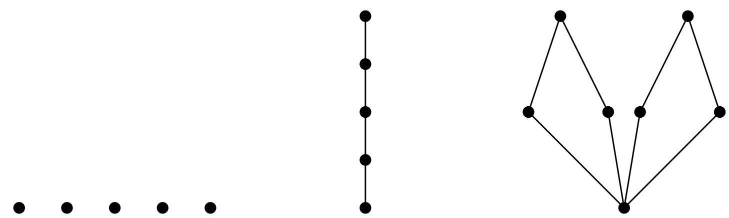
As embedding Ramsey degrees are already multiplicative and monotonous one might simply take $\log t_{\mathbf{C}}(A)$ for the entropy measure. However, this entropy measure is extremely coarse as, for example, for every finite graph G with n vertices we have that $t(G) = n!$ regardless of the structure of the graph. However, thinking of entropy of a finite structure, for example a finite simple graph, as the amount of information needed to describe the structure, we expect this magnitude to be related to the size of the automorphism group of the structure: structures with few automorphisms need complex descriptions and, thus, should have high entropy, while structures with large automorphism groups are highly symmetric



$|\mathrm{Aut}(G)| = 5!$ $\quad$ $|\mathrm{Aut}(G)| = 5!$ $\quad$ $|\mathrm{Aut}(G)| = 2$

$\log_2 \frac{n!}{|\mathrm{Aut}(G)|} = 0$ $\quad$ $\log_2 \frac{n!}{|\mathrm{Aut}(G)|} = 0$ $\quad$ $\log_2 \frac{n!}{|\mathrm{Aut}(G)|} \approx 5.91$

Figure 1: $\log_2 \tilde{t}(G)$ for three graphs



$|\mathrm{Aut}(P)| = 5!$ $\quad$ $|\mathrm{Aut}(P)| = 1$ $\quad$ $|\mathrm{Aut}(P)| = 8$

$\log_2 \frac{e(P)}{|\mathrm{Aut}(P)|} = 0$ $\quad$ $\log_2 \frac{e(P)}{|\mathrm{Aut}(P)|} = 0$ $\quad$ $\log_2 \frac{e(P)}{|\mathrm{Aut}(P)|} \approx 3.32$

Figure 2: $\log_2 \tilde{t}(P)$ for three posets

and of low entropy. Whether the automorphism group of a structure is "small" or "large" is usually measured with respect to the size of the full symmetric group. So, one might expect that the entropy of a finite structure A should be related to $\frac{n!}{|\mathrm{Aut}(A)|}$, where n is the size of A, Fig 1.

Taking the entropy of A to be simply the logarithm of this quantity does not suffice for at least two reasons: in most cases $\log \frac{n!}{|\mathrm{Aut}(A)|}$ is not monotonous in A; on the other hand, while $\log \frac{n!}{|\mathrm{Aut}(A)|}$ displays many desirable properties in case of finite simple graphs, for other combinatorially interesting classes of structures such as finite partial orders the additive property cannot be established. Interestingly, in case of finite partial orders replacing $\frac{n!}{|\mathrm{Aut}(A)|}$ with $\frac{e(A)}{|\mathrm{Aut}(A)|}$, where $e(A)$ is the number of linear extensions of A, yields a quantity possessing the same desirable properties, Fig 2. Note that $\frac{n!}{|\mathrm{Aut}(G)|}$ and $\frac{e(P)}{|\mathrm{Aut}(P)|}$ are exactly the structural Ramsey degrees of graphs, resp. posets, as computed in [3, 4, 5]. That is why our goal is to invest into building a more refined measure based on "semicontinuous regularization" of structural Ramsey degrees.

3 Results

Let $\mathbf{C}$ be a locally small category whose morphisms are mono. Fix $A, B \in \mathrm{Ob}(\mathbf{C})$ so that $A \to B$. A partition Λ of $\binom{B}{A}$ – the set of all subobjects of B isomorphic to A (see [11] for the precise definition) is *essential* if there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $B \to C$ and for every partition Π of $\binom{C}{A}$ there is a $w \in \hom(B, C)$ such that $\Lambda \succcurlyeq \ell_w^{-1}(\Pi)$. (Here, ℓ_w denotes the left multiplication by w.) Let $\mathrm{Ess}\binom{B}{A}$ be the set of all the essential partitions of $\binom{B}{A}$.

Definition 3.1. Let $\mathbf{C}$ be a small category whose morphisms are mono and homsets are finite, and let H be an entropy on partitions. Define $\tilde{r} : \mathrm{Ob}(\mathbf{C}) \to \mathbb{R} \cup \{\infty\}$ as follows:

$$\tilde{r}(X) = \inf_{A : X \to A} \sup_{B : A \to B} \min_{\Lambda \in \mathrm{Ess}\binom{B}{A}} H(\Lambda).$$

We refer to $\tilde{r}$ as the *Ramsey entropy based on H*, and say that $\mathbf{C}$ *admits the Ramsey entropy based on H* if $\tilde{r}(X) < \infty$ for all $X \in \mathrm{Ob}(\mathbf{C})$. In particular, if $\tilde{r}$ is based on the Boltzmann entropy $H^{Bol}(\Pi) = \log|\Pi|$, we refer to $\tilde{r}$ as the *Ramsey-Boltzmann entropy*.

Theorem 3.2. *Let $\mathbf{C}$ be a small category whose morphisms are mono and homsets are finite. Let H be an arbitrary entropy on partitions and let $\tilde{r}$ be the Ramsey entropy based on H.*

(a) If $X \to Y$ for some $X, Y \in \mathrm{Ob}(\mathbf{C})$ then $\tilde{r}(X) \leqslant \tilde{r}(Y)$;

(b) $\tilde{r}(X) \leqslant \log \tilde{t}(X)$ for all $X \in \mathrm{Ob}(\mathbf{C})$ (where we assume $\log \infty = \infty$).

(c) If $\mathbf{C}$ has finite structural Ramsey degrees then $\mathbf{C}$ admits the Ramsey entropy.

(d) If $X \in \mathrm{Ob}(\mathbf{C})$ is a subramsey object in $\mathbf{C}$ then $\tilde{r}(X) = 0$; in particular if X is a Ramsey object in $\mathbf{C}$ then $\tilde{r}(X) = 0$.

Theorem 3.3. *Ramsey-Boltzmann entropy is the maximal Ramsey entropy on a category. More precisely, let $\mathbf{C}$ be a small category whose morphisms are mono and homsets are finite, let $\tilde{r}'$ be a Ramsey entropy and $\tilde{r}$ the Ramsey-Boltzmann entropy. Then $\tilde{r}'(X) \leqslant \tilde{r}(X)$ for all $X \in \mathrm{Ob}(\mathbf{C})$.*

Theorem 3.4. *Let $\mathbf{C}$ be a small category with amalgamation whose morphisms are mono and homsets are finite.*

(a) $\mathbf{C}$ has finite structural Ramsey degrees if and only if $\mathbf{C}$ admits the Ramsey-Boltzmann entropy.

(b) Assume that $\mathbf{C}$ admits the Ramsey-Boltzmann entropy $\tilde{r}$. Then $X \in \mathrm{Ob}(\mathbf{C})$ is a subramsey object in $\mathbf{C}$ if and only if $\tilde{r}(X) = 0$.

Theorem 3.5. *Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be small categories with amalgamation whose morphisms are mono and homsets are finite, and let $\tilde{r}_{\mathbf{C}_1}$ and $\tilde{r}_{\mathbf{C}_2}$ be the Ramsey-Boltzmann entropies. Then for all $X_1 \in \mathrm{Ob}(\mathbf{C}_1)$ and $X_2 \in \mathrm{Ob}(\mathbf{C}_2)$ we have that $\tilde{r}_{\mathbf{C}_1 \times \mathbf{C}_2}(X_1, X_2) = \tilde{r}_{\mathbf{C}_1}(X_1) + \tilde{r}_{\mathbf{C}_2}(X_2)$.*

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