Structured Versus Decorated Cospans

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An "open system" is any sort of system that can interact with the outside world. Experience has shown that many open systems are nicely modeled using cospans. The apex of a cospan describes the system itself, while its feet describe "interfaces" through which the system can interact with the outside world, and its legs describe how the interfaces are included in the system. If a category A has finite colimits, we can compose cospans in A using pushouts: this describes the operation of attaching two open systems by identifying one interface of the first with one of the second. We can also "tensor" cospans using coproducts: this describes setting open systems side by side, in parallel. Via these operations we obtain a symmetric monoidal double category with cospans in A as horizontal 1-cells. Vertical morphisms in this double category describe maps between open systems.

However, we often want a system to have more structure than its interfaces do. This led Fong to develop "decorated" cospans [9]. Given a category A with finite colimits and a symmetric lax monoidal functor $F\colon (\mathsf{A},+) \to (\mathsf{Set},\times)$, we can equip the apex m of a cospan in A with extra structure: an element $s \in F(m)$ called a "decoration". Thus a decorated cospan is a pair of this sort:

$$a \xrightarrow{\;i\;} m \xleftarrow{\;o\;} b, \qquad\qquad s \in F(m).$$

Decorated cospans have been used to describe a variety of open systems: for example electrical circuits, Markov processes, chemical reaction networks and dynamical systems [5, 6, 8]. Unfortunately, some results in these papers were wrong. The problem is that while Fong's decorated cospans are good for decorating the apex of a cospan with an element of a set, they are unable to decorate it with an object of a category.

We can solve this problem by working instead with a symmetric lax monoidal pseudofunctor $F\colon (\mathsf{A},+) \to (\mathbf{Cat},\times)$. We have shown [4, Thm. 2.2] that this gives a symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$ in which:

- an object is an object of A,
- a vertical 1-morphism is a morphism of A,
- a horizontal 1-cell from a to b is a decorated cospan:

$$a \xrightarrow{\;i\;} m \xleftarrow{\;o\;} b, \qquad s \in F(m),$$

- a 2-morphism is a map of decorated cospans: that is, a commutative diagram

$$\begin{array}{ccccccc} a & \xrightarrow{\;i\;} & m & \xleftarrow{\;o\;} & b & & s \in F(m) \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \\ a' & \xrightarrow{\;i'\;} & m' & \xleftarrow{\;o'\;} & b' & & s' \in F(m') \end{array}$$

  together with a morphism $\tau\colon F(h)(s) \to s'$ in $F(m')$.

As detailed in [4, Sec. 6], this generalization allows us to correct all mistakes in the cited papers.

However, there is also another way to equip the apex of a cospan with extra structure. This is the theory of structured cospans [3]. Given a functor $L\colon \mathsf{A} \to \mathsf{X}$, a "structured cospan" is a cospan in X whose feet come from a pair of objects in A:

$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b).$$

When A and X have finite colimits and L preserves them, there is a symmetric monoidal double category ${}_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ where:

- an object is an object of A,
- a vertical 1-morphism is a morphism of A,
- a horizontal 1-cell from a to b is a diagram in X of this form:

$$L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$$

- a 2-morphism is a commutative diagram in X of this form:

$$\begin{array}{ccccc} L(a) & \xrightarrow{i} & x & \xleftarrow{o} & L(b) \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle L(g)} \\ L(a') & \xrightarrow[i']{} & x' & \xleftarrow[o']{} & L(b') \end{array}$$

When both structured and decorated cospans apply, structured cospans are somewhat simpler to use. This is why the latter have been implemented in code in the AlgebraicJulia project [1] and then applied to epidemiological modeling [2, 7]. But this raises an interesting mathematical question: when can decorated cospans be seen as structured cospans?

Here we provide a useful sufficient condition [4, Thm. 4.1]. Suppose A has finite colimits and $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ is a symmetric lax monoidal pseudofunctor. Then each category $F(a)$ for $a \in A$ becomes symmetric monoidal, and F can be given the structure of a pseudofunctor $F\colon \mathsf{A} \to \mathbf{SymMonCat}$. Let $\mathbf{Rex}$ be the 2-category of categories with finite colimits, functors preserving finite colimits, and natural transformations. We show that if $F\colon \mathsf{A} \to \mathbf{SymMonCat}$ factors through $\mathbf{Rex}$ as a pseudofunctor, the decorated cospan double category $F\mathbb{C}\mathbf{sp}$ is equivalent to a structured cospan double category, which we can describe explicitly. In fact, they are isomorphic as symmetric monoidal double categories.

Thus, under these conditions the decorated and structured cospan frameworks are equivalent. But there also important examples in which these conditions do not hold, such as open dynamical systems [4, Sec. 6.4]. Thus, there continue to be interesting open questions about the relation between the two formalisms.

References

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