Monoidal Reverse Differential Categories
Extended Abstract

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Cartesian reverse differential categories (CRDCs) are a recently defined structure which categorically model the reverse differentiation operations used in supervised learning. Here we define a related structure known as a *monoidal reverse differential category* and prove important results about its relationship to CRDCs. This is an extended abstract for our preprint: arXiv:2203.12478 [4].

To handle notions of differentiation that have become more prominent in computer science, two categorical structures have been useful: monoidal differential categories (MDC) [1] and Cartesian differential categories (CDC) [2]. Each axiomatizes a different aspect of differentiation: MDCs axiomatize the linear maps and then derive the smooth maps from them; conversely, CDCs axiomatize the smooth maps and derive the linear maps from them. While these structures have been very useful, they both only represent the "forward" aspect of differentiation. For uses of the derivative in supervised learning, the "reverse" derivative is typically much more useful.

Thus, a natural question is how to modify MDCs and CDCs to handle reverse differentiation. For the Cartesian side of the picture, this was already accomplished in [3]. While a CDC involves a category which comes equipped with an operator D which for any map $f : A \to B$ outputs a map $\mathsf{D}[f] : A \times A \to B$, a Cartesian *reverse* differential category (CRDC) comes equipped with an operator R which for any map $f : A \to B$ outputs a map $\mathsf{R}[f] : A \times B \to A$. It was shown in [3] that a CRDC can be seen as a CDC with additional structure. Specifically, a CRDC is equivalent to giving a CDC in which the subcategory of linear maps has a transpose operator, which categorically speaking is a special type of of dagger structure. The explicit connection with supervised learning was then made in [5], which showed how to describe several supervised-learning techniques in the abstract setting of a CRDC.

However, the first CRDC paper [3] left open the question of what a *monoidal* reverse differential category (MRDC) should be. The goal of this project is to fill in this gap by defining *monoidal reverse differential categories* (MRDC) and establishing their fundamental relationships to the existing categorical differential structures described above.

	Cartesian	Monoidal
Forward	CDC [2]	MDC [1]
Reverse	CRDC [3]	MRDC (this project)

What should this structure look like? As mentioned above, CDCs axiomatize smooth maps, while MDCs axiomatize linear maps. However, as noted above, for a CRDC, its subcategory of linear maps has dagger structure. So at a minimum, an MRDC should have dagger structure. However, we argue that an MRDC should be even stronger: it should be *self-dual compact closed*. Why do we ask for this additional structure? There are two important requirements we ask of an MRDC.

1. Just as every CRDC gives a CDC so should a MRDC give a MDC; moreover, we should be able to characterize precisely what structure is required of an MDC to make it an MRDC (as we can in the Cartesian case [3]).
2. Just as the coKleisli category of an MDC is a CDC [2], so should the coKleisli category of an MRDC be a CRDC.

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These requirements force an MRDC to be self-dual compact closed.

To prove these results, it will be helpful to investigate the basic structure of an MDC more closely. In particular, we add a new aspect to the story of an MDC: a "context fibration" which helps to relate the structure of MDCs to CDCs (and then similarly between MRDCs and CRDCs).

Thus, the main contributions of this project are as follows:

1. Give the basic definition of a MRDC, along with examples, including some unexpected ones in quantum computation, specifically related to Selinger and Valiron's programming language for quantum computation [6] and Vicary's categorical quantum harmonic oscillators [7].

2. Prove theorems that describe the relationships of MRDCs to CDCs, CRDCs, and MDCs.

3. Add some additional material about the relationship of MDCS to CDCs via a "context fibration".

MRDCs thus provide the possibility of combining linear logic and/or quantum computation with supervised learning, which is an exciting direction we hope will be pursued in the near future.

References

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