Lax Liftings and Lax Distributive Laws

Ezra Schoen
University of Strathclyde
Glasgow, Scotland

Liftings of endofunctors on sets to endofunctors on relations are commonly used to capture bisimulation of coalgebras. Lax versions have been used in those cases where strict lifting fails to capture bisimilarity, as well as in modeling other notions of simulation. This paper provides tools for defining and manipulating lax liftings.

As a central result, we define a notion of a lax distributive law of a functor over the powerset monad, and show that there is an isomorphism between the lattice of lax liftings and the lattice of lax distributive laws.

We also study two functors in detail: (i) we show that the lifting for monotone bisimilarity is the minimal lifting for the monotone neighbourhood functor, and (ii) we show that the lattic of liftings for the (ordinary) neighbourhood functor is isomorphic to P(4).

1 Introduction

Coalgebras for an endofunctor are a general model of state-based transition systems. [9]. Bisimulations are a central concept in the study of coalgebras, describing behavioral equivalence of states. Going back to [14], bisimulations of F-coalgebras in **Sets** have been defined as prefixed points of $\bar{F}$, the extension of F to **Rel**, the category of sets and relations.

One issue is that **Rel** places high demands on extensions: if $\widetilde{F} : \mathbf{Rel} \to \mathbf{Rel}$ is to be a strict functor that preserves the ordering of relations, and coincides with F on graphs of functions, then $\widetilde{F}$ only exists if F preserves weak pullbacks[5]; and if F preserves them, it is unique [4] and equal to the Barr lifting $\bar{F}$ [2]. This situation is undesireable for two reasons:

- The elegant extension-based framework for bisimulation cannot be directly applied to coalgebras of type F when F does not preserve weak pullbacks. Neighbourhood-type functors are the most prominent example of such F.
- While the lifting $\bar{F}$ can be used to reason about bisimulation, other notions of simulation or equivalence of coalgebras cannot be expressed in the same way, since there are no other strict extensions.

To remedy this, various weaker notions of extension have been proposed.[18] [1][8][12]. Finding explicit examples has proceeded in a mostly ad-hoc fashion. The aim of this paper is to provide tools to reason about lax lifting in a more principled way. This paper is based on chapter 3 of the author's MSc thesis [16].

Our main contribution is a new notion of a *lax distributive law*, which we will show are in one-to-one correspondence with lax liftings. Distributive laws at their most general are simply natural transformations $FG \Rightarrow GF$ for two functors F, G. In most cases however, at least one of the two functors F and G is taken to be a monad, and the distributive law is required to interact 'nicely' with the monad structure.

The connection between liftings and distributive laws originates in [3], which focused monad-monad interactions. Mulry [13] proved the equivalence between distributive laws of a functor F over a monad T and liftings of F to the Kleisli category of T.

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More recently, some notions of 'weak distributive law' have been studied [17]; these, like Beck, pertain to monad-monad interaction, and involve weakening some of the conditions on Becks original distributive laws. Closer to the work in this paper are the lax distributive laws in [19], though again these focus on monad-monad interactions.

Aside from their connection to monads, distributive laws are of interest in their own right. They feature centrally in the bialgebraic approach to operational semantics [20][10]. In the theory of automata, morphisms of distributive laws can provide various determinization procedures. [22]

We also analyse the liftings for two specific functors in detail:

- We prove that the minimal lifting for the monotone neighbourhood functor is given by the lifting $\widetilde{\mathscr{M}}$. This lifting has previously been used [15]; our result shows that $\widetilde{\mathscr{M}}$ is in some sense universal for $\mathscr{M}$.
- We give a complete description of the liftings for the ordinary neighbourhood functor. Equivalence notions between neighbourhood structures can be quite complex. [7] The classification in this paper shows that any notion of bisimulation between neighbourhood structures based on lax liftings will be almost trivial, since none of the 16 possible liftings makes meaningful use of the input relation.

Outline

In section 2, we show that for a fixed functor, the lax liftings form a complete lattice. This implies that any functor admits a minimal, "maximally expressive" lifting. We show that for weak pullback-preserving functors, the minimal lifting coincides with the Barr lifting.

In section 3, we define lax distributive laws, and show that there is an isomorphism between the lattice of lax liftings, and the lattice of lax distributive laws. We also characterize those distributive laws that correspond to liftings that are symmetric and diagonal-preserving.

In section 4, we study the monotone and ordinary neighbourhood functors in more detail. For the monotone neighbourhood functor, we show that the known lifting $\widetilde{\mathscr{M}}$ is minimal. For the ordinary neighbourhood functor, we show that the lattice of liftings is isomorphic to $P(4)$ by giving an explicit description of all 16 liftings.

2 Preliminaries and basic properties

Definition 1. We write **Rel** for the category of sets and relations. The objects of **Rel** are sets, and a morphism $R \in \mathbf{Hom}_{\mathbf{Rel}}(X,Y)$ is given by a subset $R \subseteq X \times Y$.

Given two relations $R : X \multimap Y$ and $S : Y \multimap Z$, we write $R;S : X \multimap Z$ for their composition $R;S = \{(x,z) \in X \times Z \mid \exists y : xRySz\}$. Note that the order of composition is reversed from function composition.

Given a relation $R : X \multimap Y$, we write R° for its converse; that is,

$$R^\circ = \{(y,x) \mid (x,y) \in R\}$$

Given a function $f : X \to Y$, we write $\mathrm{gr}(f)$ for its *graph*, which is the relation

$$\mathrm{gr}(f) = \{(x,y) \mid f(x) = y\}$$

The category **Rel** is enriched over posets, where relations are ordered by inclusion. This makes **Rel** into a 2-category (in fact, it is the canonical example of an allegory). The operation $(-)^\circ$ is the morphism part of a functor $(-)^\circ : \mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$, which is an isomorphism of 2-categories. We write $\mathrm{gr}^\circ : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Rel}$ for the composition $(-)^\circ \circ \mathrm{gr}$.

Remark 2. The category **Rel** is isomorphic to the Kleisli category for the powerset monad. The assignment $f \mapsto \mathrm{gr}(f)$ is the morphism part of the left adjoint gr in the free-forgetful adjunction $\mathrm{gr} \dashv P$ that arises out of the Kleisli category construction. For a given relation $R : X \multimap Y$, we will write $\chi_R : X \to PY$ for the corresponding Kleisli morphism. Conversely, for a Kleisli morphism $f : X \to PY$, we will write $\lfloor f \rfloor : X \multimap Y$ for the corresponding relation.

The converse of a Kleisli morphism $f : X \to PY$ will be written as

$$f^\flat : Y \to PX : y \mapsto \{x \mid f(x) \ni y\}$$

Definition 3. Let $F : \mathbf{Sets} \to \mathbf{Sets}$ be a functor. A *(lax) F-lifting* is a lax 2-functor $L : \mathbf{Rel} \to \mathbf{Rel}$ such that

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{L} & \mathbf{Rel} \\ {\scriptstyle\mathrm{gr}}\uparrow & \leq & \uparrow{\scriptstyle\mathrm{gr}} \\ \mathbf{Sets} & \xrightarrow{F} & \mathbf{Sets} \end{array} \qquad \begin{array}{ccc} \mathbf{Rel} & \xrightarrow{L} & \mathbf{Rel} \\ {\scriptstyle\mathrm{gr}^\circ}\uparrow & \leq & \uparrow{\scriptstyle\mathrm{gr}^\circ} \\ \mathbf{Sets}^{\mathrm{op}} & \xrightarrow{F^{\mathrm{op}}} & \mathbf{Sets}^{\mathrm{op}} \end{array}$$

commute up to the indicated inequalities. A lifting L is called *symmetric* if

$$\begin{array}{ccc} \mathbf{Rel}^{\mathrm{op}} & \xrightarrow{L^{\mathrm{op}}} & \mathbf{Rel}^{\mathrm{op}} \\ {\scriptstyle(-)^\circ}\uparrow & & \uparrow{\scriptstyle(-)^\circ} \\ \mathbf{Rel} & \xrightarrow{L} & \mathbf{Rel} \end{array}$$

commutes; it is called *diagonal-preserving* if it strictly preserves identities.

Explicitly, we can expand the above into the following 5 conditions:

1. (2-cells) For all $R, S : X \multimap Y$, if $R \leq S$, then $LR \leq LS$.

2. (lax functoriality) For all $R : X \multimap Y$ and $S : Y \multimap Z$, we have $LR;LS \leq L(R;S)$.

3. (lifting) For all $f : X \to Y$, we have

$$\mathrm{gr}(Ff) \leq L\,\mathrm{gr}(f), \qquad \mathrm{gr}^\circ(Ff) \leq (L\,\mathrm{gr}(f))^\circ$$

4. (diagonal-preserving) For all X, we have $L\Delta_X \leq \Delta_{FX}$.

5. (symmetry) For all $R : X \multimap Y$, we have

$$L(R^\circ) = (LR)^\circ$$

Remark 4. The above includes various notions of lifting that have been previously been studied. Some authors (e.g. [11]) have taken "lifting" to be synonymous with the Barr lifting (see below). The notion of "(weak) relator" in [1] and [18] strengthen condition 2 to strict functoriality (although [18] does not require monotonicity). The notion used in [12] is almost identical, the only difference being that they require symmetry.

We give some examples:

Example 5. (5.1) For all functors $F : \mathbf{Sets} \to \mathbf{Sets}$, there is the lifting $F_\top : \mathbf{Rel} \to \mathbf{Rel}$ given by

$$F_\top(R : X \multimap Y) = FX \times FY$$

This lifting is symmetric, but does not preserve diagonals unless $|FX| \leq 1$ for all X.

(5.2) Any relation $R : X \multimap Y$ is presented as a span $R = \mathrm{gr}^\circ(\pi_1^R); \mathrm{gr}(\pi_2^R)$ by the two projection functions $\pi_1^R : R \to X$ and $\pi_2^R : R \to Y$. This motivates the definition

$$\bar{F}X = \mathrm{gr}^\circ(F\pi_1); \mathrm{gr}(F\pi_2)$$

$\bar{F}$ is known as the *Barr lifting*; it originates in [2]. In general, $\bar{F}$ is not lax but oplax, meaning $LR; LS \geq L(R; S)$. However, if F preserves weak pullbacks, then $\bar{F}$ is a strict functor which strictly preserves graphs and converse graphs.[11] Since the diagonal is the graph of the identity, $\bar{F}$ also preserves diagonals.

(5.3) The *Neighborhood functor* is defined to be the functor $\mathscr{N} = PP$. The action on a morphism $f : X \to Y$ is given by

$$(\mathscr{N} f)U = \{v \mid f^{-1}(v) \in U\}$$

The *Monotone neighborhood functor* is the subfunctor $\mathscr{M}$ of $\mathscr{N}$ defined by

$$\mathscr{M}X = \{U \in \mathscr{N}X \mid u \in U \text{ and } u \subseteq u' \implies u' \in U\}$$

One lifting for the monotone neighbood functor is given by

$$\widetilde{\mathscr{M}}(R : X \multimap Y) = \{(U,V) \mid \forall u \in U \exists v \in V : \forall y \in v \exists x \in u : xRy \text{ and } \forall v \in V \exists u \in U : \forall x \in u \exists y \in v : xRy\}$$

This lifting originates in [15] where it was used to prove uniform interpolation for monotone modal logic. A closely related notion of bisimulation appeared earlier in [6].

We also state a simple lemma on lax liftings:

Lemma 6. *Let L be an F-lifting. For all relations $R : X \multimap Y$ and all functions $f : X' \to X$ and $g : Y' \to Y$, we have*

$$L(\mathrm{gr}(f); R; \mathrm{gr}^\circ(g)) = \mathrm{gr}(Ff); LR; \mathrm{gr}^\circ(Fg)$$

This is lemma 3.10(iii) in [16].

For a given functor $F : \mathbf{Sets} \to \mathbf{Sets}$, write $\mathbf{Lift}(F) = \{L : \mathbf{Rel} \to \mathbf{Rel} \mid L \text{ is an } F\text{-lifting}\}$. Liftings are naturally ordered pointwise: we say $L \leq L'$ if and only if for all R, we have $LR \leq L'R$.

Theorem 7. *Fix a functor $F : \mathbf{Sets} \to \mathbf{Sets}$. The class $\mathbf{Lift}(F)$ forms a complete lattice, with meets given by*

$$\left(\bigwedge_{i \in I} L_i\right) R := \bigcap_{i \in I} (L_i R)$$

Proof. See appendix. □

Since complete lattices have a minimal element, we get the following corollary:

Corollary 8. *Every endofunctor on* **Sets** *admits a minimal lifting.*

The significance of this corollary is the following: each lifting gives rise to a corresponding notion of simulation of coalgebras, as well as a modal logic. If for two liftings L, L' we have $L \leq L'$, then L-simulation distinguishes more states than L'-simulation, and L-logic is more expressive than L'-logic. A minimal lifting hence induces a maximally discerning notion of (bi)simulation, and a maximally expressive logic (among those that arise from lax liftings). [16]

In case F is weak pullback-preserving, we have an explicit description of its minimal lifting.

Proposition 9. *Let F : **Sets** $\to$ **Sets** be weak pullback-preserving. Then $\bar{F}$ is minimal among the F-liftings.*

Proof. Let L be a lifting for F. Then let $R : X \multimap Y$ be a relation. We know that R is presented as a span $R = \mathrm{gr}^\circ(\pi_X^R); \mathrm{gr}(\pi_Y^R)$ with $\pi_X^R : R \to X$ and $\pi_Y^R : R \to Y$ being the projection functions. So,

$$LR = L(\mathrm{gr}^\circ(\pi_X^R); \mathrm{gr}(\pi_Y^R)) \geq L(\mathrm{gr}^\circ(\pi_X^R)); L(\mathrm{gr}(\pi_Y^R)) \geq \mathrm{gr}^\circ(F\pi_X^R); \mathrm{gr}(F\pi_Y^R) = \bar{F}R$$

□

There is also a natural involution on liftings, induced by $(-)^\circ$:

Definition 10. For an F-lifting L, we define the lifting $L^\sim$ as

$$L^\sim(R) := (L(R^\circ))^\circ$$

It is simple to prove that $L^\sim$ is a lifting when L is. [16]

Natural transformations between functors also induce a map between the associated liftings:

Theorem 11. *Let F, G : **Sets** $\to$ **Sets** be functors, and let $\eta : F \Rightarrow G$ be a natural transformation.*

(i) For every G-lifting L, the assignment

$$R \mapsto \{(x,y) \in FX \times FY \mid (\eta(x), \eta(y)) \in LR\}$$

constitutes an F-lifting $\eta^* L$.

(ii) η^* preserves arbitrary meets and $(-)^\sim$.

(iii) If L is symmetric, so is $\eta^* L$.

(iv) If η is everywhere injective, then if L preserves diagonals, so does $\eta^* L$.

Note that joins are not preserved in general: in particular, the minimal lifting is rarely preserved by η^*.

Proof. See appendix. □

From point (iv), together with the fact that the Barr lifting always preserves diagonals, we immediately get the following result:

Corollary 12. *All subfunctors of a weak pullback-preserving functor admit a diagonal-preserving lifting.*

This motivates the following conjecture:

Conjecture 13. The converse of the above: if F has a diagonal-preserving lifting, it can be embedded in a weak pullback-preserving functor.

3 Lax distributive laws

In this section, we give an alternative characterization of relation lifting in terms of distributive laws. We will write $\mu : P^2 \to P$ and $\eta : \mathrm{id} \to P$ for respectively the multiplication and unit of the powerset monad.

Definition 14. Let $F : \mathbf{Sets} \to \mathbf{Sets}$ be any functor. A *lax distributive law for F* is a collection of maps $\lambda_{-} : FP(-) \to PF(-)$, satisfying:

(Monotonicity) For any two functions $f, g : X \to PY$, if $f \leq g$, then

$$\lambda_Y \circ Ff \leq \lambda_Y \circ Fg$$

(Weak naturality) For any function $f : X \to PY$, we have

$$PFf \circ \lambda_X \leq \lambda_{PY} \circ FPf$$

(Weak monadicity) For any Z, we have

$$\mu_{FZ} \circ P\lambda_Z \circ \lambda_{PZ} \leq \lambda_Z \circ F\mu_Z \text{ and } \lambda_Z \circ F\eta_Z \geq \eta_{FZ}$$

There are also the optional properties

(Weak extensionality) For any Z,

$$\lambda_Z \circ F\eta_Z \leq \eta_{FZ}$$

(Symmetry) For any map $f : X \to PY$,

$$(\lambda_Y \circ Ff)^\flat = \lambda_X \circ F(f^\flat)$$

Definition 15. Let $\lambda : FP \rightsquigarrow PF$ be a lax distributive law. For a given relation $R : X \multimap Y$, we define $L^\lambda R$ as

$$L^\lambda R := \lfloor \lambda_Y \circ F\chi_R \rfloor$$

Conversely, for a lax lifting L of F, we define $\lambda^L : FP \rightsquigarrow PF$ as

$$\lambda^L := \chi_{L\ni}$$

It is not yet clear that these operations do result in a distributive law and a F-lifting respectively. This will be the main theorem of this section.

Theorem 16. *Let $F : \mathbf{Sets} \to \mathbf{Sets}$ be a functor.*

(i) If L is a F-lifting, then λ^L is a lax distributive law. Moreover, if L preserves diagonals then λ^L is weakly extensional, and if L is symmetric, then λ^L is symmetric.

(ii) If λ is a lax distributive law, then L^λ is a F-lifting. Moreover, if λ is weakly extensional, then L^λ preserves diagonals, and if λ is symmetric, then L^λ is symmetric.

(iii) The operations $L \mapsto \lambda^L$ and $\lambda \mapsto L^\lambda$ are inverse to each other.

Proof. (i) We check the conditions in order.

(Monotonicity) We see that

$$\lfloor \lambda^L_Y \circ Ff \rfloor = \lfloor \lambda^L_Y \circ F\chi_{\lfloor f \rfloor} \rfloor = L^{\lambda^L}(\lfloor f \rfloor) = L(\lfloor f \rfloor)$$

where we already use point (iii) for the final equality. Now monotonicity of λ^L follows immediately from monotonicity of L.

(Weak naturality) Note that

$$(\ni_X; \mathrm{gr}(f)) \subseteq (\mathrm{gr}(Pf); \ni_{PY})$$

since if $A \ni x$, then $Pf[A] \ni f(x)$.

Now we see that

$$\begin{aligned}
a \in PFf \circ \lambda^L_X(\Phi) &\iff \exists a' : a = Ff(a') \text{ and } a' \in \lambda^L_X(\Phi) \\
&\iff \exists a' : a = Ff(a') \text{ and } a' \in \chi_{L\ni}(\Phi) \\
&\iff \exists a' : a = Ff(a') \text{ and } (\Phi, a') \in L(\ni_X) \\
&\iff (\Phi, a) \in L(\ni_X); \mathrm{gr}(Ff) \\
&\implies (\Phi, a) \in L(\ni_X; \mathrm{gr}(f)) \\
&\implies (\Phi, a) \in L(\mathrm{gr}(Pf); \ni_{PY}) \\
&\iff (\Phi, a) \in \mathrm{gr}(FPf); L(\ni_{PY}) \\
&\iff a \in \lambda^L_{PY} FPf(\Phi)
\end{aligned}$$

(Weak monadicity) First, we write out that

$$\mu \circ P\lambda^L_Z \circ \lambda^L_{PZ} = \mu \circ P(\chi_{L\ni}) \circ \chi_{L\ni} = \chi_{L\ni;L\ni}$$

since χ_{-} turns relational composition ; into Kleisli composition. Next, note that

$$\mathrm{gr}(\mu); \ni = \ni; \ni$$

since

$$\bigcup_{A \in \mathscr{A}} A \ni x \text{ if and only if } \exists A : \mathscr{A} \ni A \text{ and } A \ni x$$

So, we conclude that

$$\begin{aligned}
\lfloor \mu \circ P\lambda^L_Z \circ \lambda^L_{PZ} \rfloor &= L\ni; L\ni \\
&\leq L(\ni; \ni) \\
&= L(\mathrm{gr}(\mu); \ni) \\
&= \mathrm{gr}(F\mu); L\ni \\
&= \lfloor \mu \circ P\chi_{L\ni} \circ \eta \circ F\mu \rfloor \\
&= \lfloor \chi_{L\ni} \circ F\mu \rfloor \\
&= \lfloor \lambda^L_Z \circ F\mu \rfloor
\end{aligned}$$

giving the first inequality.

For the second inequality, we simply note that $\lfloor \eta_Z \rfloor = \mathrm{gr}(\mathrm{id}_Z)$, and so

$$\lfloor \lambda^L_Z \circ F\eta_Z \rfloor = L^{\lambda^L} \lfloor \eta_Z \rfloor = L\,\mathrm{gr}(\mathrm{id}_Z) \geq \mathrm{gr}(F\,\mathrm{id}_Z) = \mathrm{gr}(\mathrm{id}_{FZ}) = \lfloor \eta_{FZ} \rfloor$$

(Weak extensionality) Assume that L is diagonal-preserving. We aim to show that λ_L is weakly extensional. This follows simply from

$$\lfloor \lambda^L_Z \circ F\eta_Z \rfloor = L\,\mathrm{gr}(\mathrm{id}_Z) \leq \mathrm{gr}(\mathrm{id}_{FZ}) = \lfloor \eta_{FZ} \rfloor$$

(Symmetry) If L is symmetrical, we get simply

$$\lfloor (\lambda^L_Y \circ Ff)^\flat \rfloor = (L\lfloor f \rfloor)^\circ = L(\lfloor f \rfloor^\circ) = \lfloor \lambda_X \circ F(f^\flat) \rfloor$$

(ii) We prove each of the five conditions.

(2-cells) If $S \leq R$, then

$$L^\lambda S = \lfloor \lambda_Y \circ F\chi_S \rfloor \leq \lfloor \lambda_Y \circ F\chi_R \rfloor = L^\lambda R$$

by monotonicity of λ.

(lax functoriality) Let $R : X \multimap Y$ and $S : Y \multimap Z$. We draw the following diagram:

$$\begin{array}{ccccc}
FX & \xrightarrow{F(\chi_{R;S})} & FPZ & \xrightarrow{\lambda_Z} & PFZ \\
\downarrow{\scriptstyle F\chi_R} & = & \uparrow{\scriptstyle F\mu_Z} & \geq & \uparrow{\scriptstyle \mu_{FZ}} \\
FPY & \xrightarrow{FP\chi_S} & FPPZ & \xrightarrow{P\lambda_Z \circ \lambda_{PZ}} & PPFZ \\
\downarrow{\scriptstyle \lambda_Y} & \leq & \downarrow{\scriptstyle \lambda_{PZ}} & = & \| \\
PFY & \xrightarrow{PF\chi_S} & PFPZ & \xrightarrow{P\lambda_Z} & PPFZ
\end{array}$$

The top left square is F applied to the Kleisli composite $\chi_{R;S}$. The top right square is weak monadicity, and the bottom left square is weak naturality. The bottom right square is a simple equality.

The above diagram shows that

$$L^\lambda R; L^\lambda S = \lfloor \mu_{FZ} \circ P(\lambda_Z \circ F\chi_S) \circ \lambda_Y \circ F\chi_R \rfloor \leq \lfloor \lambda_Z \circ F(\chi_{R;S}) \rfloor = L^\lambda(R;S)$$

as desired.

(lifting) Let $f : X \to Y$ be a morphism. Then

$$\begin{aligned}
L^\lambda\,\mathrm{gr}(f) &= L^\lambda(\lfloor \eta_Y \circ f \rfloor) \\
&= \lfloor \lambda_Y \circ F(\eta_Y \circ f) \rfloor \\
&= \lfloor \lambda_Y \circ F\eta_Y \circ Ff \rfloor \\
&\geq \lfloor \eta_{TY} \circ Ff \rfloor = \mathrm{gr}(Ff)
\end{aligned}$$

by weak monadicity. We also have

$$\begin{aligned}
\mathrm{gr}(Ff); L^\lambda\,\mathrm{gr}^\circ(f) &= \lfloor \mu_X \circ P\lambda_X \circ PF(\chi_{\mathrm{gr}^\circ(f)}) \circ \eta_{FX} \circ Ff \rfloor \\
&= \lfloor \lambda_X \circ F(\chi_{\mathrm{gr}^\circ(f)}) \circ Ff \rfloor \\
&= \lfloor \lambda_X \circ F(\chi_{\mathrm{gr}^\circ(f)} \circ Ff) \rfloor \\
&\geq \lfloor \lambda_X \circ F\eta_X \rfloor \\
&\geq \lfloor \eta_{FX} \rfloor = \Delta_{FX}
\end{aligned}$$

and since $\mathrm{gr}^\circ(Ff)$ is the least relation R with $\mathrm{gr}(Ff);R \geq \Delta_X$, we obtain

$$L^\lambda \,\mathrm{gr}^\circ(f) \geq \mathrm{gr}^\circ(Ff)$$

as desired.

(diagonal-preserving) Assume that λ is weakly extensional. Then

$$L^\lambda \Delta_Z = \lfloor \lambda_Z \circ \eta_Z \rfloor \leq \lfloor \eta_{FZ} \rfloor = \Delta_{FZ}$$

(symmetry) Assume that λ is symmetric. Then it follows immediately that

$$L^\lambda(R^\circ) = \lfloor \lambda_X \circ F(\chi_R^\flat) \rfloor = \lfloor (\lambda_Y \circ F\chi_R)^\flat \rfloor = \lfloor \lambda_Y \circ F\chi_R \rfloor^\circ = (L^\lambda R)^\circ$$

(iii) We calculate

$$\lfloor \lambda_Z^{L^\lambda} \rfloor = L^\lambda(\ni_Z) = \lfloor \lambda_Z \circ F(\ni_Z) \rfloor = \lfloor \lambda_Z \circ F\chi_\ni \rfloor = \lfloor \lambda_Z \circ F\,\mathrm{id}_{PZ} \rfloor = \lfloor \lambda_Z \rfloor$$

showing $\lambda^{L^\lambda} = \lambda$.

For the other equality, we get

$$\begin{aligned} L^{\lambda^L}(R) &= \lfloor \lambda^L \circ F\chi_R \rfloor \\ &= \lfloor \chi_{L\ni} \circ F\chi_R \rfloor \\ &= \lfloor \mu \circ P\chi_{L\ni} \circ \eta \circ F\chi_R \rfloor \\ &= \lfloor F\chi_R \rfloor ; \lfloor \chi_{L(\ni)} \rfloor \\ &= \mathrm{gr}(F\chi_R); L\ni \\ &= L(\mathrm{gr}(\chi_R);\ni) \\ &= LR \end{aligned}$$

□

4 Explicit descriptions

Since the class of F-liftings forms a complete lattice for each F, it follows that each F has a minimal lifting $\tilde{F}$. In the case of weak-pullback preserving F, we know that $\tilde{F} = \bar{F}$, the Barr lifting. However, for non-weak-pullback preserving functors, giving an explicit description of the minimal lifting involves a non-trivial amount of effort.

In this section, we will study the minimal liftings for the neighborhood functor and the monotone neighborhood functor. For the (ordinary) neighborhood functor, we moreover give a full description of the complete lattice of liftings.

4.1 Monotone neighborhood functor

Recall the lifting $\widetilde{\mathscr{M}}$ from example (5.3).

Theorem 17. *The lifting $\widetilde{\mathscr{M}}$ is the minimal lifting for the monotone neighborhood functor $\mathscr{M}$.*

To prove this, we first need a lemma.

Lemma 18. *Let $R : X \multimap Y$ be a total surjective relation. Then $\widetilde{\mathscr{M}}R \le LR$ for all liftings L.*

In [6], a similar statement appears as lemma 4.7.

Proof. Consider the two projection morphisms $\pi_X : R \to X$ and $\pi_Y : R \to Y$. Since R is total and surjective, both these functions are surjective.

We claim that $\widetilde{\mathscr{M}}R = (\mathscr{M}\,\pi_X)^\circ;\mathscr{M}\,\pi_Y$. The inequality $\ge$ follows from $R = (\pi_X)^\circ;\pi_Y$.

For $\le$, let $(U,V) \in \widetilde{\mathscr{M}}R$. Then we set

$$\begin{aligned} W_0 &:= \{\{(x,y) \in R \mid x \in u\} \mid u \in U\} \\ W_1 &:= \{\{(x,y) \in R \mid y \in V\} \mid v \in V\} \\ W &:= \{w \mid \exists w' \in W_0 \cup W_1 : w' \subseteq w\} \end{aligned}$$

We claim that $\mathscr{M}\,\pi_X(W) = U$. For this, we need to show that (1) if $u \in U$, then $\pi_X^{-1}(u) \in W$, and (2) if $\pi_X^{-1}(u) \in W$, then $u \in U$.

(1) Clearly, if $u \in U$, then $\pi_X^{-1}(u) = \{(x,y) \in R \mid x \in u\} \in W$, so $\pi_X^{-1}(u) \in W$.

(2) Assume $\pi_X^{-1}(u) \in W$. There are two cases: (i) there is a $u' \in U$ with $\{(x,y) \in R \mid x \in u'\} \subseteq \pi_X^{-1}(u)$, or (ii) there is a $v \in V$ with $\{(x,y) \in R \mid y \in v'\} \subseteq \pi_X^{-1}(u)$.

(i) In this case, we know that $\pi_X[\{(x,y) \in R \mid x \in u'\}] \subseteq \pi_X(\pi_X^{-1}(u))$. But since R was total, we know that $\pi_X[\{(x,y) \in R \mid x \in u'\}] = u'$ and $\pi_X[\pi^{-1}(u)] = u$. So $u' \subseteq u$, and hence $u \in U$.

(ii) Clearly, $\pi_X[\{(x,y) \in R \mid y \in v\}] = \{x \mid \exists y \in v : xRy\}$. Since $(U,V) \in \widetilde{\mathscr{M}}R$, there is a $u' \in U$ such that for all $x \in u'$, there is a $y \in v$ with xRy. But this just says that $u' \subseteq \pi_X[\{(x,y) \in R \mid y \in v\}]$. So we conclude that there is a $u' \in U$ with

$$u' \subseteq \pi_X[\{(x,y) \in R \mid y \in v\}] \subseteq \pi_X(\pi_X^{-1}(u)) = u$$

and hence $u \in U$.

So in both cases, we have $u \in U$, as desired.

The proof that $\mathscr{M}\,\pi_Y(W) = V$ is completely symmetrical; so, we can conclude that $(U,V) \in (\mathscr{M}\,\pi_X)^\circ;\mathscr{M}\,\pi_Y$.

Now, let L be any lifting. Then

$$LR = L((\pi_X)^\circ;\pi_Y) \ge L(\pi_X)^\circ;L\pi_Y \ge (\mathscr{M}\,\pi_X)^\circ;\mathscr{M}\,\pi_Y = \widetilde{\mathscr{M}}R$$

□

With this lemma, we can prove theorem 17.

Proof. Let $R : X \multimap Y$ be any relation. Let X' be the domain of R and Y' the range of R. Then we define $X_* = X \cup \{*\}, Y_* = Y \cup \{*\}$ and

$$R_* = R \cup \{(x,*) \mid x \in X \setminus X'\} \cup \{(*,y) \mid y \in Y \setminus Y'\} \cup \{(*,*)\}$$

Then $R_* : X_* \multimap Y_*$ is total and surjective.

Let $\iota_X : X \to X_*$ and $\iota_Y : Y \to Y_*$ be the natural inclusion functions. First, we note that $R = \iota_X;R_*;(\iota_Y)^\circ$ The inequality $\le$ is clear, since $R \subseteq R_*$. For $\ge$, notice that $*$ is not in the range of either ι_X or ι_Y.

Now by lemma 6, we know that for any lifting L,

$$LR = (\mathscr{M}\,\iota_X);LR_*;(\mathscr{M}\,\iota_Y)^\circ.$$

So we can calculate that

$$\begin{aligned} LR &= \mathscr{M}\,\iota_X;LR_*;(\mathscr{M}\,\iota_Y)^\circ \\ &\geq \mathscr{M}\,\iota_X;\widetilde{\mathscr{M}}R_*;(\mathscr{M}\,\iota_Y)^\circ && \text{by lemma 18} \\ &= \widetilde{\mathscr{M}}R \end{aligned}$$

We conclude that $\widetilde{\mathscr{M}}$ is minimal. □

4.2 The neighborhood functor

We introduce an extremely minimal logic for neighborhood systems. This will consist of the following expressions:

$$\begin{aligned} \rho_0 &::= \Box\bot \mid \neg\Box\bot \\ \rho_1 &::= \Box\top \mid \neg\Box\top \\ \rho &::= (\rho_0, \rho_1) \end{aligned}$$

Given $(U,V) \in \mathscr{N}X \times \mathscr{N}Y$, satisfaction $(U,V) \Vdash \rho$ is defined as follows:

$$\begin{aligned} &(U,V) \Vdash \Box\bot \text{ iff } \varnothing \in U \implies \varnothing \in V \\ &(U,V) \Vdash \neg\Box\bot \text{ iff } \varnothing \notin U \implies \varnothing \notin V \\ &(U,V) \Vdash \Box\top \text{ iff } X \in U \implies Y \in V \\ &(U,V) \Vdash \neg\Box\top \text{ iff } X \notin U \implies Y \notin V \\ &(U,V) \Vdash (\rho_0,\rho_1) \text{ iff } (U,V) \Vdash \rho_0 \text{ and } (U,V) \Vdash \rho_1 \end{aligned}$$

Now let I be the set of all ρ's. For each $J \subseteq I$, we get a lifting L_J defined on a relation $R : X \multimap Y$ as

$$L_J(R) := \{(U,V) \in \mathscr{N}X \times \mathscr{N}Y \mid (U,V) \Vdash \rho \text{ for all } \rho \in J\}$$

Since these liftings do not depend on the chosen relation, we will omit R, writing simply $L_J : \mathscr{N}X \multimap \mathscr{N}Y$. Note also that $J \supseteq J'$ if and only if $L_J \leq L_{J'}$.

Theorem 19. *The lattice $(P(I), \supseteq)$ is isomorphic to $(\mathbf{Lift}(\mathscr{N}), \leq)$ via $J \mapsto L_J$.*

To prove this theorem, we will need the following lemma:

Lemma 20. *Let $(U,V) \in \mathscr{N}X \times \mathscr{N}Y$ and $(U',V') \in \mathscr{N}X' \times \mathscr{N}Y'$. Assume that for some ρ, we have $(U,V) \nVdash \rho$ and $(U',V') \nVdash \rho$. Then for each ρ', we have*

$$(U,U') \Vdash \rho', \quad (V,V') \Vdash \rho'$$

Proof. WLOG, we can assume that $\rho = (\Box\bot, \Box\top)$; all other cases are similar.

Then since $(U,V) \nVdash \rho$, we know $\varnothing \in U, \varnothing \notin V$ and $X \in U, Y \notin V$. Similarly, we know $\varnothing \in U', \varnothing \notin V$ and $X' \in U', Y' \notin V'$. But from these data, it follows immediately that for all ρ', we must have

$$(U,U') \Vdash \rho'$$

since U and U' agree on $\varnothing$ and the entire set. And of course the same holds for (V,V').

□

Now we can start the full proof.

Proof. First, we show that each L_J is a lifting. Since clearly $L_J = \bigwedge_{\rho \in J} L_{\{\rho\}}$, it suffices to show that each $L_{\{\rho\}}$ is a lifting.

They are clearly monotonic, since they do not depend on the input R. They are also clearly laxly functorial. Finally, if $f : X \to Y$ is a function, then for all $U \in \mathscr{N} X$ and all $\rho \in I$, we have

$$(U, (\mathscr{N} f)U) \Vdash \rho$$

since

$$(\mathscr{N} f)U \ni \varnothing \text{ iff } U \ni f^{-1}(\varnothing) \text{ iff } U \ni \varnothing \text{ and } (\mathscr{N} f) \ni X \text{ iff } U \ni f^{-1}(X) \text{ iff } U \ni Y$$

So indeed, each L_J extends the graph of $\mathscr{N} f$.

This shows that the map $J \mapsto L_J$ is well-defined. It is clearly injective and meet-preserving (recall that the meet in $(P(I), \supseteq)$ is given by union), so it remains to show that it is surjective. We will proceed in three steps:

1. The top element is preserved by $J \mapsto L_J$;
2. The bottom element is preserved by $J \mapsto L_J$;
3. If $L > L_J$, then there is some $J' \subsetneq J$ with $L \geq L_{J'}$.

These three steps together imply that $J \mapsto L_J$ is surjective, from which it then follows that it is an isomorphism.

For point 1: The top element of $(P(I), \supseteq)$ is $\varnothing$, and indeed $L_\varnothing(R : X \multimap Y) = X \times Y$.

For point 2: Let L be any symmetric lifting for $\mathscr{N}$. For a given X, write $0_X : X \multimap X$ for the empty relation. We will show that $(U,V) \in L0_X$ if U and V agree on $\varnothing$ and X.

We first assume that X contains some point x_0. Write $2 = \{a, b\}$ for the generic two-element set; by abuse of notation, we may also consider $a, b : X \to 2$ and $x_0 : X \to X$ as constant maps.

Let $U \in \mathscr{N} X$ be a neighborhood system. There are four cases:

(i) $\varnothing \notin U, X \notin U$. Then we see that

$$\mathscr{N} a(U) = \varnothing = \mathscr{N} b(U)$$

since for constant maps $c : X \to Y$, we have $c^{-1}(A) = \varnothing$ or $c^{-1}(A) = X$ for all A. We also clearly have $\mathscr{N} b(\varnothing) = \varnothing$. So, we have

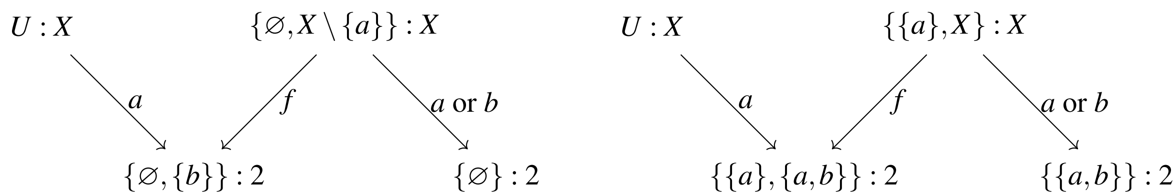
$$(U, \varnothing) \in \mathrm{gr}(\mathscr{N} a), (\varnothing, U) \in \mathrm{gr}^\circ(\mathscr{N} b)$$

for all U omitting $\varnothing$ and X. Now we have if U, V both omit $\varnothing$ and X, then

$$U(L\mathrm{gr}(a))\varnothing(L\mathrm{gr}^\circ(b))V$$

and hence

$$(U,V) \in L\mathrm{gr}(a); L\mathrm{gr}^\circ(b) \subseteq L0_X$$



(a) (b)

(ii) $\varnothing \notin U, X \in U$. Then $\mathscr{N} a(U) = \{\varnothing, \{b\}\}$. Take $f : X \to 2$ given by

$$f(x) = \begin{cases} b & x \neq x_0 \\ a & x = x_0 \end{cases}$$

Let $V = \{\varnothing, X \setminus \{x_0\}\}$. Then it is easily seen that $\mathscr{N} f(V) = \{\varnothing, \{b\}\}$. Finally, we have $\mathscr{N} a(V) = \mathscr{N} b(V) = \{\varnothing\}$, again by the remarks on inverse images along constant maps.

Now we have a 'zigzag' as in figure 1a. By tracing the definitions, we can see that $\mathrm{gr}(a); \mathrm{gr}^\circ(f) = \mathrm{gr}(x_0)$, and

$$\mathrm{gr}(a); \mathrm{gr}^\circ(f); \mathrm{gr}(a) = \mathrm{gr}(x_0); \mathrm{gr}(a) = \mathrm{gr}(a)$$

and similarly $\mathrm{gr}(a); \mathrm{gr}^\circ(f); \mathrm{gr}(b) = \mathrm{gr}(b)$. We now have that if U is such that $\varnothing \in U, X \notin U$, then

$$(U, \{\varnothing\}) \in L(\mathrm{gr}(a)), \text{ and } (\{\varnothing\}, U) \in L(\mathrm{gr}^\circ(b))$$

But now for all U, V which both contain $\varnothing$ and both omit X, we have

$$(U, V) \in L\mathrm{gr}(a); L(\mathrm{gr}^\circ(b)) \subseteq L(\mathrm{gr}(a); \mathrm{gr}^\circ(b)) = L0_X$$

(iii) Let U be such that $\varnothing \notin U, X \in U$. Then $\mathscr{N} a(U) = \{\{a\}, \{a, b\}\}$. Take $V = \{\{x_0\}, X\}$. Then with f as in point (ii), we have $\mathscr{N} f(V) = \{\{a\}, \{a, b\}\}$. Now for the constant maps a, b, we have $\mathscr{N} a(V) = \{\{a, b\}\} = \mathscr{N} b(V)$. Hence, we obtain a similar zigzag as in point (ii), as can be seen in figure 1b. From here, the argument is completely the same as in (ii): for U, V both omitting $\varnothing$ and both including X, we get

$$(U, \{\{a, b\}\}) \in L\mathrm{gr}(a), \qquad (\{\{a, b\}\}, V) \in L\mathrm{gr}^\circ(b)$$

showing that

$$(U, V) \in L(\mathrm{gr}(a); (\mathrm{gr}(b))^\circ) = L0_X$$

(iv) $\varnothing \in U, X \in U$. Then $\mathscr{N} a(U) = P2 = \mathscr{N} b(U)$, and so as in (i) we get for all U, V both including $\varnothing$ and X that

$$(U, P2) \in L\mathrm{gr}(a), \qquad (P2, V) \in L\mathrm{gr}^\circ(b)$$

and hence

$$(U, V) \in L(\mathrm{gr}(a); \mathrm{gr}^\circ(b)) = L0_X$$

Now we have that if X is nonempty, then $L0_X \supseteq L_I$. But of course, if X and Y are arbitrary, then the empty relation $0_{XY} : X \multimap Y$ factors through 0_{X+Y} via the inclusions $\iota_X : X \to X + Y, \iota_Y : Y \to X + Y$. From this, it follows easily that $L0_{XY} \supseteq L_I$. But now, for $R : X \multimap Y$ an arbitrary relation, we have that

$$LR \supseteq L0_{XY} \supseteq L_I$$

showing that L_I is minimal indeed.

For point 3: Let L be any lifting, and $J \subseteq I$ with $L > L_J$. Then there is some relation $R : X \multimap Y$ and some neighborhood systems $(U,V) \in \mathscr{N}X \times \mathscr{N}Y$ with $(U,V) \in LR$ and $(U,V) \nVdash \rho_0$ for some $\rho_0 \in J$.

We claim that now for $J' = J \setminus \{\rho_0\}$, we have $L \geq L_{J'}$. Again, we will show that $L0_{X'Y'} \geq L_{J'}$ for all X', Y'.

Now let $(U',V') \in L_{J'}$. There are two cases:

(i) $(U',V') \Vdash \rho_0$. Then $(U',V') \in L_J < L$, so $(U',V') \in L0_{X'Y'}$.

(ii) $(U',V') \nVdash \rho_0$. Since $(U,V) \nVdash \rho_0$ we know by lemma 20 that

$$(U',U) \in L_I, \quad (V,V') \in L_I$$

and hence

$$(U',V') \in L_I;LR;L_I \subseteq L0_{X'X};LR;L_{YY'} \subseteq L(0_{X'X};R;0_{YY'}) = L0_{X'Y'}$$

So indeed, $L0_{X'Y'} \geq L_{J'}$ and hence for arbitrary relations $R' : X' \multimap Y'$ we have

$$LR \geq L0_{X'Y'} \geq L_{J'}$$

as desired.

□

5 Conclusion and further research

We have shown that for a fixed functor, the lax liftings form a complete lattice. In particular, any functor admits a minimal, "maximally expressive" lifting. We show that for weak pullback-preserving functors, the least functor coincides with the Barr lifting.

We have defined lax distributive laws, and show that there is an isomorphism between the lattice of lax liftings, and the lattice of lax distributive laws. We also characterize those distributive laws that correspond to liftings that are symmetric and diagonal-preserving.

We studied the monotone and ordinary neighbourhood functors in more detail. For the monotone neighbourhood functor, we show that the known lifting $\widetilde{\mathscr{M}}$ is minimal. For the ordinary neighbourhood functor, we have explicitly described all 16 liftings. This question was still open in [16].

The results in this paper are specific to the categories of **Sets** and 2-valued relations. Other kinds of liftings have been considered. For instance, in [21], liftings of fuzzy relations are defined. A natural direction of further research is to investigate if the results from this paper could be extended to cover a wider range of many-valued relations. More generally still, one can see **Sets** as the category of functions inside the allegory **Rel**. A possible approach would be to study liftings in the setting of arbitrary (power) allegories.

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A Additional proofs

Proof of theorem 7. We show that $\mathbf{Lift}(F)$ has all meets. Let $\{L_i \mid i \in I\}$ be any collection of F-liftings. For a given $R : X \multimap Y$, set

$$LR = \bigcap_{i\in I} L_i R$$

We show that L is again a lifting, by showing it satisfies conditions 1, 2 and 3.

(1) If $R \leq S$, then

$$LR = \bigcap_{i\in I} L_i R \leq \bigcap_{i\in I} L_i S = LS$$

(2) If $R : X \multimap Y$ and $S : Y \multimap Z$ are relations, then

$$\begin{aligned} LR;LS &= \left(\bigcap_{i\in I} L_i R\right);\left(\bigcap_{i\in I} L_i S\right) \\ &\leq \bigcap_{i\in I}\bigcap_{j\in I} L_i R; L_j S \\ &\leq \bigcap_{i\in I} L_i R; L_i S \\ &\leq \bigcap_{i\in I} L_i(R;S) \\ &= L(R;S) \end{aligned}$$

(3) If $f : X \to Y$ is a function, then

$$L\,\mathrm{gr}(f) = \bigcap_{i\in I} L_i\,\mathrm{gr}(f) \geq \bigcap_{i\in I} \mathrm{gr}(Ff) = \mathrm{gr}(Ff).$$

The other inequality is similar.

So L is a lifting, and is clearly the greatest lower bound for the L_i.

□

Proof of theorem 11. (i) We check the three conditions.

(2-cells) If $R \leq R'$, then

$$\eta^* L(R) = (\eta \times \eta)^{-1}(LR) \leq (\eta \times \eta)^{-1}(LR') = \eta^* L(R')$$

since for any function f, we know that f^{-1} preserves inclusions.

(lax functoriality) If $R : X \multimap Y$ and $S : Y \multimap Z$, we have

$$\begin{aligned}\eta^* L(R;S) &= \{(x,z) \mid (\eta(x),\eta(z)) \in L(R;S)\} \\ &\geq \{(x,z) \mid \eta(x,z) \in LR;LS\} \\ &\geq \{(x,z) \mid \exists y \in Y : (\eta(x),\eta(y)) \in LR, (\eta(y),\eta(z)) \in LS\} \\ &= \eta^* L(R);\eta^* L(S)\end{aligned}$$

(lifting) Let $f : X \to Y$ be a function. Naturality of η states that $\mathrm{gr}(Ff);\mathrm{gr}(\eta) = \mathrm{gr}(\eta);\mathrm{gr}(Gf)$. From this, it follows that $\mathrm{gr}(Ff) \leq \mathrm{gr}(\eta);\mathrm{gr}(Gf);\mathrm{gr}^\circ(\eta)$. Hence, we have

$$\mathrm{gr}(Ff) \leq \mathrm{gr}(\eta);\mathrm{gr}(Gf);\mathrm{gr}^\circ(\eta) \leq \mathrm{gr}(\eta);L\,\mathrm{gr}(f);\mathrm{gr}^\circ(\eta) = \eta^* L(\mathrm{gr}(f))$$

and

$$\mathrm{gr}^\circ(Ff) \leq \mathrm{gr}(\eta);\mathrm{gr}^\circ(Gf);\mathrm{gr}^\circ(\eta) \leq \mathrm{gr}(\eta);L(\mathrm{gr}^\circ(f));\mathrm{gr}^\circ(\eta) = \eta^* L(\mathrm{gr}^\circ(f))$$

(ii) For meets, we have

$$\eta^*(\bigwedge_i L_i)(R) = (\eta \times \eta)^{-1}(\bigcap_i (L_i R)) = \bigcap_i (\eta \times \eta)^{-1}(L_i R) = \left(\bigwedge_i L_i\right)(R)$$

since meets are preserved by inverse images. For $(-)^\sim$, we have

$$\begin{aligned}\eta^*(L^\sim)(R) &= (\eta \times \eta)^{-1}(L^\sim R) \\ &= (\eta \times \eta)^{-1}((L(R^\circ))^\circ) \\ &= ((\eta \times \eta)^{-1}(L(R^\circ)))^\circ \\ &= (\eta^* L(R^\circ))^\circ \\ &= (\eta^* L)^\sim(R)\end{aligned}$$

(iii) This follows directly from preservation of $(-)^\sim$: we have

$$\begin{aligned}L \text{ is symmetric} &\iff L = L^\sim \\ &\implies \eta^* L = \eta^*(L^\sim) \\ &\iff \eta^* L = (^\sim \eta^* L) \\ &\iff \eta^* L \text{ is symmetric}\end{aligned}$$

(iv) Assume η is everywhere injective, and L preserves diagonals. Then let X be arbitrary. For all $(x,y) \in FX \times FX$, we have

$$\begin{aligned}(x,y) \in \eta^* L\Delta_X &\iff (\eta(x),\eta(y)) \in L\Delta_X \\ &\implies \eta(x) = \eta(y) && \text{since } L \text{ preserves diagonals} \\ &\implies x = y && \text{since } \eta \text{ is injective}\end{aligned}$$

and hence $\eta^* L$ preserves diagonals.

□