Simplicial distributions and quantum contextuality

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In the past couple of decades several different approaches have been developed to study the phenomenon of quantum contextuality [1–4]. Among these the sheaf-theoretic approach has yielded a successful framework for a systematic study of contextuality for nonsignaling distributions arising from general sets of commuting quantum observables. More recently, a topological approach was introduced in [5] that is well-suited for studying contextuality as a computational resource in measurement-based quantum computation [6]. Building off of these efforts in this paper (see [7] for the arXiv version) we introduce a new framework that subsumes both the topological and sheaf-theoretic approaches, based on the theory of simplicial sets, which are combinatorial models of topological spaces generalizing simplicial complexes.

Contextuality in our approach is defined for scenarios consisting of *spaces of measurements* and *spaces of outcomes*. Such spaces are modeled by *simplicial sets* [8]: objects consisting of a sequence of sets specifying simplices, together with the *simplicial relations* which give instructions on how they are *glued* together. In categorical terms a simplicial set is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ from the opposite of the simplex category[1] to the category of sets. There is a category **sSet** of simplicial sets whose morphisms are given by natural transformations of functors. Ordinary or *discrete* measurement scenarios specified by a *set* of measurements and outcomes can be embedded into this framework by regarding each element as a zero-dimensional simplex of a simplicial set. In this paper we will instead focus on realizations where measurements and outcomes label higher dimensional simplices. The increase in dimension as compared to the discrete case facilitates a richer connection between topology and contextuality (see Fig. (1a)), with important results being proved as a consequence.

Modeling measurements and outcomes as simplicial sets allows us to generalize the types of distributions available in our framework. More formally, a *simplicial scenario* is a pair (X, Y) consisting of a space X of measurements and a space Y of outcomes. We define a *simplicial distribution* on a scenario (X, Y) to be a map of simplicial sets (briefly, a map of spaces) $p : X \to D_RY$, where D_RY is a space of distributions given by a simplicial set whose simplicies are given by R-distributions [9] on the simplices of the outcome space. In other words, the simplicial set D_RY is obtained as the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{Y} \mathbf{Set} \xrightarrow{D_R} \mathbf{Set}$.

To define contextuality in our framework, let us first introduce the notion of outcome assignments, which are represented by space maps $r : X \to Y$ where Y is an arbitrary space of outcomes and for each such map there is a *deterministic distribution* denoted by $\delta^r : X \to D_RY$. A *classical distribution* is an R-convex combination of deterministic distributions. Let us write $\mathrm{S}(X, Y)$ for the set of simplicial distributions, i.e. the set $\mathbf{sSet}(X, D_RY)$ of morphisms in the category of simplicial

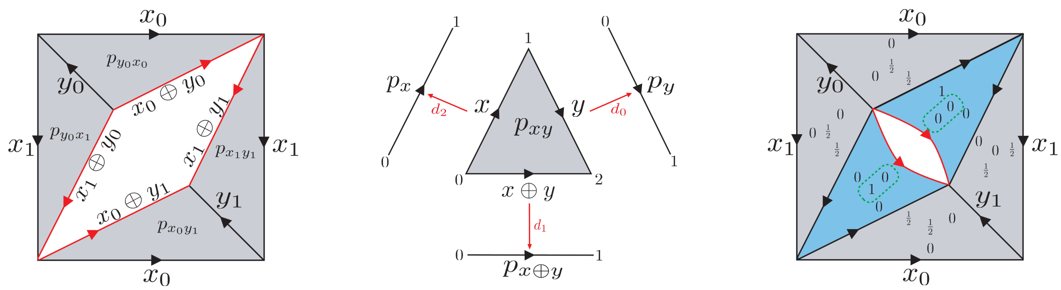
[1] The objects of the simplex category are given by the finite sets $[n] = \{0, 1, \cdots, n\}$ for $n \geq 0$ and morphisms of the category are given by order preserving functions.

sets, and $\mathrm{C}(X,Y)$ for the set $D_R(\mathbf{sSet}(X,Y))$ of classical distributions. There is a canonical map that sends a classical distribution to the associated simplicial distribution obtained by marginalizing distributions on each simplex:

$$\Theta : \mathrm{C}(X,Y) \to \mathrm{S}(X,Y),$$

i.e. a natural map $D_R(\mathbf{sSet}(X,Y)) \to \mathbf{sSet}(X, D_R Y)$. A simplicial distribution $p \in \mathrm{S}(X,Y)$ is called *contextual* if it does not lie in the image of Θ. Otherwise, it is called *noncontextual*. This definition subsumes the notion of contextuality for nonsignaling distributions. In fact, ordinary nonsignaling distributions defined on discrete scenarios can be studied as a special case, but with extra freedom provided by topology. For example, the well-known Clauser, Horne, Shimony, Holt (CHSH) scenario [10], in which Alice and Bob each perform two measurements, denoted by $\{x_0, x_1\}$ and $\{y_0, y_1\}$, with corresponding outcomes in $\mathbb{Z}_2 = \{0,1\}$ can be represented as in Fig. (1a).

Our constructions are natural with respect to change of measurement spaces (and similarly outcome spaces). More precisely, a map of spaces $f : Z \to X$ induces a map $f^* : \mathrm{S}(X,Y) \to \mathrm{S}(Z,Y)$ between the simplicial distributions (and similarly between the classical distributions). This map allows us to compare contextual properties of scenarios and provide an *extension* criterion to characterize contextuality. For the CHSH scenario the realization given in Fig. (1a) characterizes contextuality in this way; for instance, the contextual Popescu-Rohrlich (PR) box [11] in Fig. (1c) fails to extend from the punctured torus to the torus.



(a) (b) (c)

Figure 1: (a) CHSH scenario organized into a surface, which is topologically equivalent to a punctured torus. The edges labeled by x_0 (and x_1) are identified. (b) On each triangle the probability distribution associated to a pair of measurements is represented by a triangle and the face maps d_i encode the marginalization to the measurements on the edges. (c) PR box fails to extend to the whole torus, characterizing it as contextual, since the marginals at the inner edges do not match: $0+1 \neq 0+0$.

The construction used for a space of distributions associated to a space of outcomes can also be carried over to quantum measurements. The *space of quantum measurements*, denoted by $Q_{\mathcal{H}}Y$, is the simplicial set whose set of simplices consists of quantum measurements (i.e. POVMs) on the Hilbert space $\mathcal{H}$ with outcomes given by the simplices of Y. A *simplicial quantum measurement* defined on a scenario (X,Y) is a map $P : X \to Q_{\mathcal{H}}Y$ of spaces. Now quantum contextuality can be introduced by generalizing the Born rule to the simplicial setting: A quantum state specified by a density operator ρ induces a map of spaces $\rho_* : Q_{\mathcal{H}}Y \to D_{\mathbb{R}_{\geq 0}}Y$. Then ρ is called *(non)contextual with respect to* P if the simplicial distribution $X \xrightarrow{P} Q_{\mathcal{H}}Y \xrightarrow{\rho_*} D_{\mathbb{R}_{\geq 0}}Y$ is (non)contextual. With these definitions the Mermin square scenario, which was studied from a cohomological perspective in [5], can also be studied using a cohomology witness for strong contextuality, constructed in [7]. Finally two foundational results from quantum theory, Gleason's theorem [12] and Kochen–Specker theorem [13], can be presented in our simplicial framework demonstrating the power of the simplicial language.

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