Dialectica: fibrations and logical principles

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Gödel's Dialectica interpretation [2] was conceived as a tool to obtain the consistency of Peano arithmetic via a proof of consistency of Heyting arithmetic. In recent years, several proof theoretic transformations, based on Gödel's Dialectica interpretation, have been used systematically to extract new content from classical proofs, following Kreisel's suggestion. This way the interpretation has found new relevant applications in several areas of mathematics and computer science.

The Dialectica interpretation has been explained in categorical terms a few times. In her doctoral work, de Paiva introduced the notion of a *Dialectica category* as an internal version of Gödel's Dialectica interpretation [1]. This was generalised by Hyland [4] and Hofstra [3], who considered the interpretation in terms of fibrations.

Building on Hofstra's work, in previous work we introduced the notions of a Gödel fibration [6], and its proof-irrelevant version, a Gödel doctrine [7]. Our presentation of the Dialectica interpretation via Gödel fibrations offers a new perspective on the Dialectica that makes it easy to prove the validity of all logical principles involved in the categorical interpretation. Also this makes it is easier to study abstract properties of the Dialectica fibration itself. The key idea is that Gödel fibrations can be thought of as fibrations generated by *quantifier-free elements* [6]. This categorification of quantifier-free elements is crucial to showing that our notion of Gödel fibration is equivalent to Hofstra's Dialectica fibration in the appropriate way, as done in [6]. But the quantifier-free elements are also crucial to show how Gödel doctrines embody the main logical features of the Dialectica Interpretation, as is done in [7].

We show that Gödel fibrations are equivalent to Hofstra's Dialectica fibrations by showing that any fibration can be seen as a Hofstra Dialectica fibration if and only if it is equivalent to a Gödel fibration [6], theorem 4.6. In the follow-up work [7], and in its extended version [8], we consider simplified, proof-irrelevant Gödel doctrines, instead of fibrations. Using Gödel doctrines, instead of fibrations, allows us to explain how the algebraic structures can be translated into the logical

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principles, which are easier to grasp and reason about. Using the explanation of the principles, we derive the soundness of the interpretation of the implication connective, as expounded by Troelstra, but this time in the categorical model. Thus we prove that Gödel doctrines satisfy the equivalence:

$$(\exists u.\forall x.A_D(u,x) \to \exists v.\forall y.B_D(v,y)) \leftrightarrow \exists f_0, f_1.\forall u,y.(A_D(u, f_1(u,y)) \to B_D(f_0(u),y))$$

for every A_D and B_D quantifier-free predicates, theorem 5.11 of [8]. This requires proving the validity of extra logical principles, going beyond intuitionistic logic, namely Markov Principle (MP) and the Independence of Premise (IP) principle, as well as a version of the axiom of choice. We show that Gödel doctrines satisfy these principles, establishing a tight (internal language) correspondence between the logical system and the categorical framework [7, 8].

In this talk we unite these two branches of previous work, reminding the reader that despite the fact that Lawvere showed us how to model first-order logic in the sixties, no one had provided a categorical description of quantifier-free objects, so far. And quantifier-free formulae are ubiquitous in logic. While the work on Gödel doctrines can be seen as an instantiation of the work on Gödel fibrations, the poset setting allows us to provide calculations using the internal language, harder to do in the fibrational setting.

We expect that this tight correspondence between the Dialectica interpretation and the categorical structure of the Gödel fibrations and doctrines will be useful not only when discussing the traditional proof-theoretical applications of the Dialectica, but also when dealing with some newer uses of the interpretation, as in modelling games or abstract machines [9, 5].

References

[1] de Paiva, V. (1989). The dialectica categories. *Categories in Computer Science and Logic*, 92:47–62.

[2] Gödel, K. (1958). Über eine bisher noch nicht benützte erweiterung des finiten standpunktes. *Dialectica*, 12(3-4):280–287.

[3] Hofstra, P. (2011). The dialectica monad and its cousins. *Models, logics, and higherdimensional categories: A tribute to the work of Mihály Makkai*, 53:107–139.

[4] Hyland, J. (2002). Proof theory in the abstract. *Annals of Pure and Applied Logic*, 114(1):43 – 78. Troelstra Festschrift.

[5] Pédrot, P.-M. (2014). A functional functional interpretation. In *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS).*

[6] Trotta, D., Spadetto, M., and de Paiva, V. (2021). The Gödel Fibration. In Bonchi, F. and Puglisi, S. J., editors, *46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021)*, volume 202 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 87:1–87:16, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[7] Trotta, D., Spadetto, M., and de Paiva, V. (2022a). Dialectica logical principles. In Artemov, S. and Nerode, A., editors, *Logical Foundations of Computer Science*, pages 346–363, Cham. Springer International Publishing.

[8] Trotta, D., Spadetto, M., and de Paiva, V. (2022b). Dialectica Principles via Gödel Doctrines (arxiv extended version).

[9] Winskel, G. (2022). Making concurrency functional (arxiv preprint).