Central Submonads and Notions of Computation

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***Abstract*—The notion of "centre" has been introduced for many algebraic structures in mathematics. A notable example is the centre of a monoid which always determines a commutative submonoid. Monads (in category theory) can be seen as generalisations of monoids and in this paper we show how the notion of centre may be extended to strong monads acting on symmetric monoidal categories. We show that the centre of a strong monad $\mathcal{T}$, if it exists, determines a commutative submonad $\mathcal{Z}$ of $\mathcal{T}$, such that the Kleisli category of $\mathcal{Z}$ is isomorphic to the premonoidal centre (in the sense of Power and Robinson) of the Kleisli category of $\mathcal{T}$. We provide three equivalent conditions which characterise the existence of the centre of $\mathcal{T}$. While not every strong monad admits a centre, we show that every strong monad on well-known naturally occurring categories does admit a centre, thereby showing that this new notion is ubiquitous. We also provide a computational interpretation of our ideas which consists in giving a refinement of Moggi's monadic metalanguage. The added benefit is that this allows us to immediately establish a large class of contextually equivalent terms for monads that admit a non-trivial centre by simply looking at the richer syntactic structure provided by the refinement.**
This submission is an extended abstract. A preprint is available at [1].

I. Technical Summary of Contributions

a) (Commutative) Submonads: Recall that given a strong monad $(\mathcal{T}, \eta, \mu, \tau)$ on a symmetric monoidal category $(\mathbf{C}, I, \otimes)$, a costrength τ' for $\mathcal{T}$ can be defined in a canonical way using the symmetry of $\mathbf{C}$. Then the monad $\mathcal{T}$ is said to be *commutative* if the following diagram:

$$
\begin{array}{ccccc}
\mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau_{\mathcal{T}X,Y}} & \mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow{\mathcal{T}\tau'_{X,Y}} & \mathcal{T}^2(X \otimes Y) \\
\downarrow{\scriptstyle \tau'_{X,\mathcal{T}Y}} & & & & \downarrow{\scriptstyle \mu_{X\otimes Y}} \\
\mathcal{T}(X \otimes \mathcal{T}Y) & \xrightarrow[\mathcal{T}\tau_{X,Y}]{} & \mathcal{T}^2(X \otimes Y) & \xrightarrow[\mu_{X\otimes Y}]{} & \mathcal{T}(X \otimes Y)
\end{array} \tag{1}
$$

commutes for every choice of objects X and Y of $\mathbf{C}$. A morphism between two strong monads $\mathcal{T}$ and $\mathcal{P}$ is a natural transformation $\iota\colon \mathcal{T} \Rightarrow \mathcal{P}$ that satisfies some obvious coherence conditions (see [2]). We say that a strong monad $\mathcal{T}$ is a *submonad* of a strong monad $\mathcal{P}$ if there exists a *monomorphism* of strong monads $\iota\colon \mathcal{T} \Rightarrow \mathcal{P}$ (in the category of strong monads over $\mathbf{C}$ with strong monad morphisms between them). In this situation, the submonad monomorphism $\iota\colon \mathcal{T} \Rightarrow \mathcal{P}$ induces a canonical categorical embedding $\mathcal{I}\colon \mathbf{C}_{\mathcal{T}} \to \mathbf{C}_{\mathcal{P}}$ between the Kleisli categories of the two monads.

b) Premonoidal Categories: Let $\mathcal{T}$ be a strong monad on a symmetric monoidal category $(\mathbf{C}, I, \otimes)$. Then, the Kleisli category $\mathbf{C}_{\mathcal{T}}$ does *not* necessarily have a canonical monoidal structure. However, it does have a canonical *premonoidal structure* as shown by Power and Robinson [3]. In fact, they show that this premonoidal structure is monoidal iff the monad $\mathcal{T}$ is commutative. What is also interesting is that Power and Robinson introduce the notion of centre for premonoidal categories which plays an important part of its definition. If $\mathbf{D}$ is a premonoidal category, then its centre, denoted $Z(\mathbf{D})$, is a *monoidal subcategory* of $\mathbf{D}$ that has the same objects. However, given a strong monad $\mathcal{T}$ on a symmetric monoidal category $\mathbf{C}$, it does not necessarily hold that the subcategory $Z(\mathbf{C}_{\mathcal{T}})$ determines a monad over $\mathbf{C}$. Nevertheless, in this situation, the left adjoint of the Kleisli adjunction $\mathcal{J}\colon \mathbf{C} \to \mathbf{C}_{\mathcal{T}}$ always corestricts to $Z(\mathbf{C}_{\mathcal{T}})$ and we write $\hat{\mathcal{J}}\colon \mathbf{C} \to Z(\mathbf{C}_{\mathcal{T}})$ to indicate this corestriction.

c) Central Submonads acting on $\mathbf{Set}$*:* The inspiration for the construction of the central submonad (if it exists) comes from the category $\mathbf{Set}$ where it can always be defined for any monad acting on it. Given a (necessarily strong) monad $\mathcal{T}\colon \mathbf{Set} \to \mathbf{Set}$, and a set X, then we define the *centre* of $\mathcal{T}$ at X, written $\mathcal{Z}X$, to be the set

$$
\begin{aligned}
\mathcal{Z}X \stackrel{\text{def}}{=} \{t \in \mathcal{T}X \mid \forall Y \in \mathrm{Ob}(\mathbf{Set}).\forall s \in \mathcal{T}Y.& \\
\mu(\mathcal{T}\tau'(\tau(t,s))) = \mu(\mathcal{T}\tau(\tau'(t,s)))\}&
\end{aligned}
$$

and we write $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ for the indicated subset inclusion. In other words, the centre of $\mathcal{T}$ at X is the subset of $\mathcal{T}X$ which contains all monadic elements for which (1) holds.

Theorem 1. *The assignment $\mathcal{Z}(-)$ extends to a* commutative submonad *of $\mathcal{T}$ with $\iota_X : \mathcal{Z}X \subseteq \mathcal{T}X$ the required monomorphism of strong monads. Furthermore, there exists a canonical isomorphism $\mathbf{Set}_{\mathcal{Z}} \cong Z(\mathbf{Set}_{\mathcal{T}})$.*

We refer to the commutative monad determined by this theorem as *the central submonad* of $\mathcal{T}$. This theorem, together with our next example, shows that we have successfully generalised the notion of centre for monoids to monads on $\mathbf{Set}$.

Example 2. Given a monoid (M, e, m), the free monad induced by M is the monad $\mathcal{T} = (M \times -)\colon \mathbf{Set} \to \mathbf{Set}$ with unit $\eta_X :: x \mapsto (e, x)$ and monad multiplication $\mu_X :: (z, (z', x)) \mapsto (m(z, z'), x)$. Then, the central submonad $\mathcal{Z}$ of $\mathcal{T}$ is given by the commutative monad $(Z(M) \times -)\colon \mathbf{Set} \to \mathbf{Set}$, where $Z(M)$ is the centre of the monoid M and where the monad data is given by the (co)restrictions of the monad data of $\mathcal{T}$. For brevity, we omit details of the (canonically induced) strength.

d) Centralisable Monads: We saw that every monad on $\mathbf{Set}$ admits a centre that yields its central submonad, which is necessarily commutative. However, a strong monad on an arbitrary symmetric monoidal category need not admit a centre at all. Next, we present three equivalent conditions for the existence of such a centre (Theorem 7).

Definition 3 (Central Cone)**.** Let X be an object of $\mathbf{C}$. A *central cone* of $\mathcal{T}$ at X is given by a pair (Z, ι) of an object Z and a morphism $\iota : Z \to \mathcal{T}X$, such that for any object Y, the diagram:

$$\begin{array}{ccccc}
Z \otimes \mathcal{T}Y & \xrightarrow{\iota \otimes \mathcal{T}Y} & \mathcal{T}X \otimes \mathcal{T}Y & \xrightarrow{\tau'_{X,\mathcal{T}Y}} & \mathcal{T}(X \otimes \mathcal{T}Y) \\
\downarrow{\scriptstyle \iota \otimes \mathcal{T}Y} & & & & \downarrow{\scriptstyle \mathcal{T}\tau_{X,Y}} \\
\mathcal{T}X \otimes \mathcal{T}Y & & & & \mathcal{T}^2(X \otimes Y) \\
\downarrow{\scriptstyle \tau_{\mathcal{T}X,Y}} & & & & \downarrow{\scriptstyle \mu_{X \otimes Y}} \\
\mathcal{T}(\mathcal{T}X \otimes Y) & \xrightarrow[\mathcal{T}\tau'_{X,Y}]{} & \mathcal{T}^2(X \otimes Y) & \xrightarrow[\mu_{X \otimes Y}]{} & \mathcal{T}(X \otimes Y)
\end{array}$$

commutes. If (Z, ι) and (Z', ι') are two central cones of $\mathcal{T}$ at X, then a *morphism of central cones* $\varphi : (Z', \iota') \to (Z, \iota)$ is a morphism $\varphi : Z' \to Z$, such that $\iota \circ \varphi = \iota'$. A *terminal central cone* of $\mathcal{T}$ at X is a central cone (Z, ι) for $\mathcal{T}$ at X, such that for any central cone (Z', ι') of $\mathcal{T}$ at X, there exists a unique morphism of central cones $\varphi : (Z', \iota') \to (Z, \iota)$.

Definition 4 (Centralisable Monad)**.** We say that the monad $\mathcal{T}$ is *centralisable* if for any object X, a terminal central cone of $\mathcal{T}$ at X exists, which we denote by writing $(\mathcal{Z}X, \iota_X)$.

Theorem 5. *If the monad $\mathcal{T}$ is centralisable, then the assignment $\mathcal{Z}(-)$ extends to a commutative monad $(\mathcal{Z}, \eta^{\mathcal{Z}}, \mu^{\mathcal{Z}}, \tau^{\mathcal{Z}})$ on $\mathbf{C}$. Moreover, $\mathcal{Z}$ is a commutative submonad of $\mathcal{T}$ with $\iota_X : \mathcal{Z}X \to \mathcal{T}X$ giving the submonad monomorphism.*

Theorem 6. *In the situation of Theorem 5, the canonical embedding functor $\mathcal{I}\colon \mathbf{C}_{\mathcal{Z}} \to \mathbf{C}_{\mathcal{T}}$ corestricts to an isomorphism of categories $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$.*

The last two theorems show that we are justified in saying the induced submonad $\mathcal{Z}$ is *the* central submonad of $\mathcal{T}$.

Theorem 7 (Centralisability)**.** *Let $\mathcal{T}$ be a strong monad on a symmetric monoidal category $\mathbf{C}$. The following are equivalent:*

1) *For any object X of $\mathbf{C}$, $\mathcal{T}$ admits a terminal central cone at X;*
2) *There exists a commutative submonad $\mathcal{Z}$ of $\mathcal{T}$ such that the canonical embedding functor $\mathcal{I} : \mathbf{C}_{\mathcal{Z}} \to \mathbf{C}_{\mathcal{T}}$ corestricts to an isomorphism of categories $\mathbf{C}_{\mathcal{Z}} \cong Z(\mathbf{C}_{\mathcal{T}})$;*
3) *The corestriction of the Kleisli left adjoint $\mathcal{J} : \mathbf{C} \to \mathbf{C}_{\mathcal{T}}$ to the premonoidal centre $\hat{\mathcal{J}} : \mathbf{C} \to Z(\mathbf{C}_{\mathcal{T}})$ also is a left adjoint.*

Example 8. Any strong monad on $\mathbf{Set}$ is centralisable (as we already saw). The same is true for any strong monad on the categories $\mathbf{DCPO}, \mathbf{Meas}, \mathbf{Top}, \mathbf{Hilb}, \mathbf{Vect}$ and many other $\mathbf{Set}$-like categories.

In the linked preprint, we describe a strong monad which is *not* centralisable. We do this by specifically constructing a full subcategory of $\mathbf{Set}$ for this purpose. However, we are not aware of any other *naturally* occurring monad described in the literature that is not centralisable.

Example 9. The valuation monad $\mathcal{V}\colon \mathbf{DCPO} \to \mathbf{DCPO}$ is strong, but its commutativity is an open problem [4]. The central submonad of $\mathcal{V}$ is precisely the "central valuations monad" described in [5]. In fact, the latter work inspired the present paper, which may be seen as a categorical generalisation of the ideas presented in [5].

Example 10. The *unbounded* Giry monad $\mathcal{G}\colon \mathbf{Meas} \to \mathbf{Meas}$, which assigns the space of all (possibly unbounded) measures to a measurable space, is a strong monad which is *not* commutative. This monad is centralisable and its central submonad $\mathcal{Z}$ is such that $\mathcal{Z}X$ contains all discrete measures on the measurable space X (and possibly others).

e) Computational Interpretation: Finally, we provide a computational interpretation of our ideas by presenting a refinement of Moggi's metalanguage [6]. The types are extended by simply adding the $\mathcal{Z}$ unary connective that represents the central submonad of $\mathcal{T}$:

$$A, B ::= \quad 1 \mid A \times B \mid A \to B \mid \mathcal{Z}A \mid \mathcal{T}A.$$

An excerpt of the more interesting terms together with their formation rules are shown below:

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathtt{ret}_{\mathcal{Z}}\ M : \mathcal{Z}A} \qquad \frac{\Gamma \vdash M : \mathcal{Z}A \quad \Gamma, x : A \vdash N : \mathcal{Z}B}{\Gamma \vdash \mathtt{do}\ x \leftarrow_{\mathcal{Z}} M; N : \mathcal{Z}B}$$

$$\frac{\Gamma \vdash M : \mathcal{Z}A}{\Gamma \vdash \iota M : \mathcal{T}A} \qquad \frac{\Gamma \vdash M : \mathcal{T}A \quad \Gamma, x : A \vdash N : \mathcal{T}B}{\Gamma \vdash \mathtt{do}\ x \leftarrow_{\mathcal{T}} M; N : \mathcal{T}B}$$

and we omit discussing constants in this short presentation, but they can be added in principle. Note that the $\mathtt{ret}_{\mathcal{Z}}$ term is now of type $\mathcal{Z}A$ instead of $\mathcal{T}A$, because the monadic unit is central. We also add an extra term for monadic sequencing for the $\mathcal{Z}$ monad and one extra term for demoting central terms of type $\mathcal{Z}A$ to $\mathcal{T}A$ which is interpreted via the ι inclusion. It is now easy to see that this system keeps track of monadic operations that are central and so we can very easily establish a large class of contextually equivalent terms by exploiting this using standard semantic arguments. As part of future work, we will describe a sound and adequate semantics of this system (which is fairly straightforward) and we will also consider the issue of full abstraction for specific monads with non-trivial centres.

References

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