Extending resource monotones using Kan Extensions

Robin Cockett

Department of Computer Science, University of Calgary, Alberta, Canada

Institute for Quantum Science and Technology, University of Calgary, Alberta, Canada

Isabelle Jianing Geng   Carlo Maria Scandolo

Department of Mathematics & Statistics, University of Calgary, Alberta, Canada

Institute for Quantum Science and Technology, University of Calgary, Alberta, Canada

Priyaa Varshinee Srinivasan

Department of Computer Science, University of Calgary, Alberta, Canada

National Institute of Standards and Technology, Maryland, USA

In this paper we generalize the framework proposed by Gour and Tomamichel regarding extensions of monotones for resource theories. A monotone for a resource theory assigns a real number to each resource in the theory signifying the utility or the value of the resource. Gour and Tomamichel studied the problem of extending monotones using set-theoretical framework when a resource theory embeds fully and faithfully into the larger theory. One can generalize the problem of computing monotone extensions to scenarios when there exists a functorial transformation of one resource theory to another instead of just a full and faithful inclusion. In this article, we show that (point-wise) Kan extensions provide a precise categorical framework to describe and compute such extensions of monotones. To set up monontone extensions using Kan extensions, we introduce partitioned categories (pCat) as a framework for resource theories and pCat functors to formalize relationship between resource theories. We describe monotones as pCat functors into $([0,\infty],\leq)$, and describe extending monotones along any pCat functor using Kan extensions. We show how our framework works by applying it to extend entanglement monotones for bipartite pure states to bipartite mixed states, to extend classical divergences to the quantum setting, and to extend a non-uniformity monotone from classical probabilistic theory to quantum theory.

1   Introduction

Resource theories [18, 6, 9] in physics model systems in which certain operations considered to be 'free of cost' among of the set of all operations. For example, placing a glass of chilled water at room temperature warms up the water to the ambient temperature. In this context, operations that change the temperature of the water to be in equilibrium with the ambient temperature are considered to be free. In order to produce a "resourceful state" — for example, a glass of chilled water — one requires non-free transformations, such as a fridge, which consumes electricity. Resource theories have been successfully used to study, among other examples, thermodynamical systems [12, 24], entanglement [19, 15], and coherence [35].

A central question in the resource-theoretic modelling of systems is: *given two resources, is there a free transformation to convert one resource into the other?* The answer to this question imposes a preorder on resources which captures their value or usefulness. Intuitively, a resource is more valuable than another if, by possessing the former, we are given access to a larger set of resources including the latter through free transformations. This not a partial order, because their may be different resources

pCat functors	functors which preserve free transformations
Preorder collapse	a pCat functor whose codomain category is a preorder
Monotone	a pCat functor whose codomain category is $([0,\infty], \leq)$
Op-monotone	a pCat functor whose codomain category is $([0,\infty], \geq)$

Table 1: Functors for resource theories

that can be converted freely into each other. Such resources are considered equivalent. In this way, we can set up a partial order on the equivalence classes of resources. One way to define such an order is to quantify resources by introducing monotones, which are a order-preserving maps from the set of all resources into $[0,\infty]$ [9]. Monotones assign a value to resources that is compatible with the preorder, viz. with their usefulness. Monotones often have a physical meaning, such as in the resource theories of quantum thermodynamics [24], where, for systems at a fixed temperature, free energy is a monotone, and for isolated systems, entropy is the natural monotone.

Given a monotone *M* for a resource theory which embeds in a larger theory, a natural question to ask is whether the monotone M for the smaller theory can be used to quantify the resources in the larger resource theory. This question arises from the observation that that resources exclusive to the larger theory can possibly be converted to resources contained in the smaller theory, and vice versa. It turns out that one can always compute the optimal upper and lower bound for the value of every resource in the larger theory. In other words, it is possible to extend the monotone M to give optimal upper and lower bounds respectively on the value of resources in the larger theory.

In [16], Gour and Tomamichel presented a set-theoretical framework for extending monotones from a subset of resources to the entire set of resources. Given a monotone *M* over a subset of states, they compute 'minimal' and 'maximal' extensions of the monotone to the entire set of states. In this article, we show that these extensions are special cases of more general categorical concepts, called (point-wise) left and right Kan extensions [20, 5, 25, 29]. Kan extensions deal with optimally extending a functor $F : \mathbb{X} \to \mathbb{A}$ along another functor $K : \mathbb{Y} \to \mathbb{A}$ to give two functors: $\overline{F}_K : \mathbb{Y} \to A$ called the left Kan extension of *F* along *K*, and $\underline{F}_K : \mathbb{Y} \to \mathbb{A}$ called the right Kan extension of *F* along *K*. The right Kan extension can be interpreted as the most conservative extension of *F* along *K* and the left Kan extension as the most liberal extension of *F* along *K*.

We first introduce partitioned Categories (pCats) as a framework for resource theories. Partitioned categories are categories with a chosen subcategory of free transformations. The subcategory includes all the objects of the parent category, in other words, the inclusion of the subcategory into the parent category is bijective on objects. Relationships between resource theories are set up as pCat functors. In this article, since we consider monotones which are not necessarily additive, thus we do not demand symmetric monoidal structure on pCats.

Given a resource theory, the necessary and sufficient conditions for transformations of resources can be encoded as a pCat functor from the resource theory into a preorder. We call such a pCat functor as a preorder collapse. A resource monotone is a preorder collapse into $([0,\infty], \leq)$. In resource theories, contravariant rather than covariant monotones are encountered more frequently, the reason being if resource *A* can be transformed to resource *B* using only free transformation(s), then the value of *A* is considered to be at least as high as the value of *B*. We refer to a contravariant resource monotone as an op-monotone. The distinction between monotones and op-monotones is important in the computation of monotone extensions. The categorical descriptions of pCats, pCat functors, preorder collapse, monotones are discussed with various running examples in Section 3.1. Table 1, we briefly summarizes the functors of resource theories introduced in this article.

Having set up monotones as pCat functors, optimal extensions of monotones along any pCat functor are given by their left and right Kan extensions. Lemma 3.35 examines the properties of the monotone extensions thus computed, and prove that such extensions are optimal and monotonic. In Lemma 3.37, we show that extending a monotone along a full and faithful functor recovers the case described in [16] by Gour and Tomamichel. We apply the Kan extension framework for monotones to extend classical divergences to the quantum setting, to extend bipartite pure states entanglement monotone to mixed states, and to extend Shannon entropy as a measure of non-uniformity from classical probabilistic theory to quantum theory. Section 3.4 is dedicated to setting up the Kan extension framework for monotones, and to studying the extension properties and its applications.

Notation: *In this paper, we use bold letters $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{D}$ to denote categories. We use uppercase letters to denote both objects in the categories and functors between categories, whose meaning will be clear from the context. Lowercase letters f, g, h, π are reserved for maps in the categories. Let X, Y, Z be objects, and let $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ be two arrows, we denote the composition of the two arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ as fg, and similar notations apply for the composition of functors.*

2 An introduction to Kan extensions

Kan extensions [20, 5, 25, 29] are a broadly applicable notion which is quite central to category theory. Indeed, Mac Lane in his book 'Categories for working Mathematician' [25] gave the chapter on Kan extensions the title "All concepts are Kan extensions". In this section, we provide the definition of Kan extensions and discuss limits and colimits as an example of Kan extensions.

2.1 Left and right Kan extensions

We first provide the definition Kan extensions of a functor along another functor, and explain the universal properties.

Definition 2.1. Let $F : \mathbb{X} \to \mathbb{D}$ and $K : \mathbb{X} \to \mathbb{Y}$ be any two functors.

   (i) **Right Kan (minimal) extension** of F along K is a functor $\underline{F}_K : \mathbb{Y} \to \mathbb{D}$ with a natural transformation $\underline{\psi} : K\underline{F}_K \Rightarrow F$ which is universal , see Fig. 2-(a). The right Kan extension is written as $(\underline{F}_K, \underline{\psi})$.

  (ii) **Left Kan (maximal) extension** of F along K is a functor $\overline{F}_K : \mathbb{Y} \to \mathbb{D}$ with a natural transformation $\overline{\psi} : F \Rightarrow K\overline{F}_K$ which is couniversal, see Fig. 2-(b). The left Kan extension is written as $(\overline{F}_K, \overline{\psi})$.
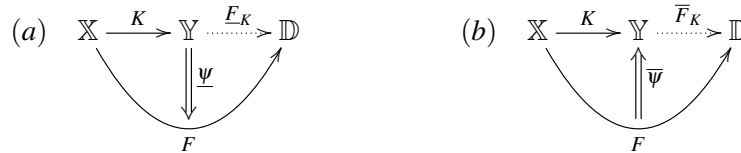

Figure 1: (a) Right Kan Extension   (b) Left Kan Extension

Fig. 1 shows the Kan extensions of F along K. We refer to the category $\mathbb{D}$ as the **target**, the category $\mathbb{X}$ as the **source categories**. Functor F is extended from its source $\mathbb{X}$ along K. Right and Left Kan extensions of F along K are usually written as $\mathrm{Ran}_K(F)$ and $\mathrm{Lan}_K(F)$. However, we use the notation introduced in [16] for resource monotone extensions for uniformity.

Let us examine the universal properties of the Kan extensions. The universal property of right Kan extension assures that for any other functor $H : \mathbb{Y} \to \mathbb{D}$ with a natural transformation $\gamma : KH \Rightarrow F$, there exists a $\gamma' : H \to \underline{F}_K$ such that γ factors through $\underline{\psi}$ via γ', that is, $\gamma = (K \otimes \gamma')\underline{\psi}$ (See Fig. 2-(a)). Informally, this means the right Kan extension of F along K is the most conservative extension and that any other extension H can be transformed to $\underline{F}_K$. In this sense, $\underline{F}_K$ is the minimal extension of F along K.

Similarly the couniversal property of left Kan extension assures that for any other functor $H : \mathbb{Y} \to \mathbb{D}$ with a natural transformation $\delta : F \Rightarrow KH$, there exists a $\delta' : \overline{F}_K \to H$ such that δ factors through $\overline{\psi}$ via δ', that is, $\gamma = \overline{\psi}(K \otimes \gamma')$ (See Fig. 2-(b)). Informally, this means that $\overline{F}_K$ can be naturally transformed to any other such H. In this sense, $\overline{F}_K$ is the maximal extension of F along K.

The universal properties of Kan extensions assure that the extensions are optimal.
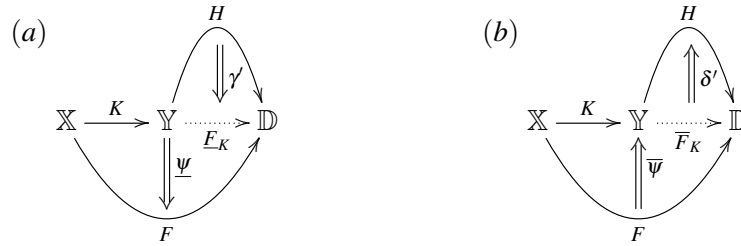

Figure 2: (a) Right Kan Extension is universal   (b) Left Kan Extension is couniversal

Example 2.2. The left and right Kan extensions of a functor F along a terminal functor $(!)$ gives precisely the limit and the colimit of F. The terminal functor maps all the objects and the maps of the domain category to the single object and the single map in the terminal category (1) respectively. Any functor proceeding from the terminal category chooses precisely one object and its identity morphism in the codomain category.
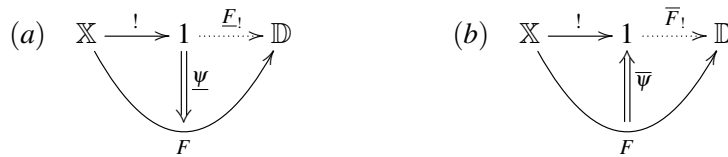
The **left Kan extension** of $F : \mathbb{X} \to 1$ along the unique functor into 1 gives a **colimiting cocone**. The functor $\underline{F}_!$ chooses precisely one object in $\mathbb{D}$ (hence we write the object as $\underline{F}_!$) which is the apex of the cocone. The natural transformation $\overline{\psi}$ has components, $\overline{\psi}_X :!F(X) \Rightarrow \underline{F}_!$ for each $X \in \mathbb{X}$.

Due to the couniversal property of $\underline{\psi}$, for any other functor $P : 1 \to \mathbb{D}$ with a natural transformation $\gamma :!P \Rightarrow F$, there exists a unique natural transformation $\gamma' : P \Rightarrow !\underline{F}_!$ such that $\gamma'\underline{\psi} = \gamma$. Hence, $\underline{F}_!$ is **the limit of diagram** F.
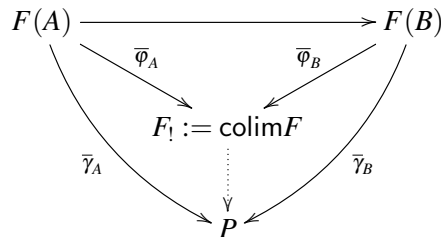
Suppose $\mathbb{D}$ is the poset $(\mathbb{R}, \le)$, then in the above diagram, $\underline{F}_!$ is precisely the **greatest lower bound** of $\{F(A), F(B)\}$.

Similarly, **the right Kan extension** gives a **limiting cone**. $\overline{F}_!$ is referred to as the **limit of diagram** F. When $\mathbb{D}$ is a poset $\overline{F}_!$ is the **least upper bound** of the subset of $\mathbb{R}$ chosen by F.

2.2   How to compute Kan extensions?

In Example 2.2, it was shown that the left and right Kan extensions of a functor F along the unique functor into the terminal category are respectively the colimit and the limit of diagram F. In this section, we show how one can compute Kan extensions of a functor when the target category is complete (has all small limits) and cocomplete (has all small colimits) and the intermediate category is locally small (the arrows between any two objects in the category is a small set).

Theorem 2.3. *[29, Thoerem 6.2.1] Given functors $F : \mathbb{X} \to \mathbb{D}$ and $K : \mathbb{X} \to \mathbb{Y}$, if the category $\mathbb{D}$ is cocomplete, then the left Kan extension $\overline{F}_K$ exists and is defined to be:*

$$\forall Y \in \mathbb{Y}, \ \overline{F}_K(Y) := \text{colim}(K \downarrow Y \xrightarrow{\pi_{K \downarrow Y}} \mathbb{X} \xrightarrow{F} \mathbb{D}) \tag{2.1}$$

with the natural transformation $\overline{\psi}$ extracted from colimiting cocones in $\mathbb{D}$.
 If $\mathbb{C}$ is complete, then the right Kan extension $\underline{F}_K$ exists and is defined to be:

$$\forall Y \in \mathbb{Y}, \ \underline{F}_K(Y) := \lim(Y \downarrow K \xrightarrow{\pi_{Y \downarrow K}} \mathbb{X} \xrightarrow{F} \mathbb{D}) \tag{2.2}$$

with the natural transformation $\underline{\psi}$ extracted from limiting cones in $\mathbb{D}$.

Proof. (Sketch)
 Suppose $F : \mathbb{X} \to \mathbb{D}$ is any functor and $\mathbb{D}$ is cocomplete. Then, one can compute the left Kan extension $(\underline{F}_K, \overline{\psi})$ of F along any functor $K : \mathbb{X} \to \mathbb{Y}$ as follows:

Defining functor $\underline{F}_K : \mathbb{Y} \to \mathbb{D}$:

The left Kan extension is computed on each point (object) in $\mathbb{Y}$.

For each object Y in $\mathbb{Y}$, consider the slice category $(K \downarrow Y)$. The objects in the slice category are pairs (X, f) where,

$$f : K(X) \to Y \in \mathbb{Y}$$

and a map $m : (X, f) \to (X, f')$ in the slice category is a map $m \in \mathbb{X}$ such that the following triangle commutes:

$$
\begin{array}{ccc}
K(X) & \xrightarrow{\ \ K(m)\ \ } & K(X') \\
 & {}_{f}\searrow \quad \swarrow{}_{f'} & \\
 & Y &
\end{array}
$$

Stated informally, the slice category contains complete information on how to arrive at an object $Y \in \mathbb{Y}$ using objects and transformations of $\mathbb{X}$. The projection functor $\pi_{K \downarrow Y}$ chooses precisely the subcategory of $\mathbb{X}$ relevant to Y, see Figure 3-(a). The left Kan extension on point Y is the colimit of the diagram F applied to this sub-category. The couniversal cocone of the diagram $\pi_{K \downarrow Y} F$ has a natural transformation $\lambda : \text{Lim}(\pi_{K \downarrow Y} F) \Rightarrow F$, with a component λ_X for each object $\pi_{K \downarrow Y}(f, K(X)) := X \in \mathbb{X}$.
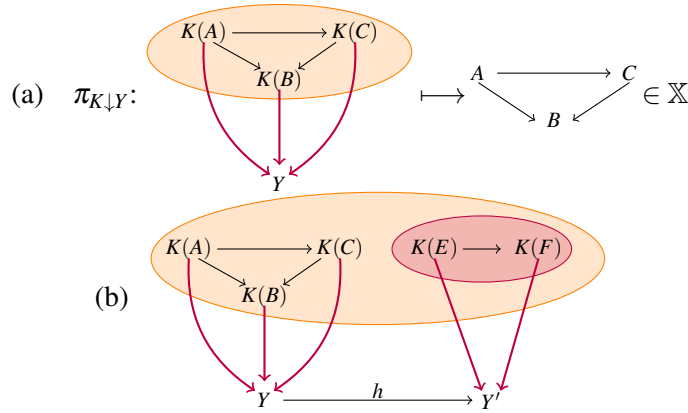 The left extension $\overline{F}_K$ is then defined as follows:

Figure 3: (a) $\pi_{K\downarrow Y}$ projects the shaded region of $(K \downarrow Y)$ into $\mathbb{X}$ (in general $K \downarrow Y$ is not a subcategory of $\mathbb{X}$); (b) An arrow $h : Y \rightarrow Y' \in \mathbb{Y}$ leads to the (shaded) base of above Y' to be included in the (shaded) base above Y. Hence the base of the colimiting cocone of $\pi_{K\downarrow Y}F$ includes the shaded base of the colimiting cone of $\pi_{K\downarrow Y}F$ inducing a unique map $\text{colim}(\pi_{K\downarrow Y}F) \rightarrow \text{colim}(\pi_{K\downarrow Y'}F)$

- For all objects $Y \in \mathbb{Y}$, $\underline{F}(Y) := \text{colim}(\pi_{K\downarrow Y}F)$.
- For all maps $h : Y \rightarrow Y'$, $\text{colim}(\pi_{K\downarrow Y}F) \rightarrow \text{colim}(\pi_{K\downarrow Y'}F)$ is the unique arrow induced by h, see Figure 3-(b).

Defining the natural transformation $\overline{\psi} : F \Rightarrow K\overline{F}_K$:

For all $X \in \mathbb{X}$, $\overline{\psi}_X$ is the component $\lim(\pi_{K\downarrow KX}F) \rightarrow F(X)$ of the colimiting cocone corresponding to the initial object $(1_K X, KX) \in (K \downarrow KX)$.

Computing right Kan extension is dual to computing left Kan extensions. If $F : \mathbb{X} \rightarrow \mathbb{D}$ is any functor and $\mathbb{D}$ is complete (contains all small limits), then one can compute the right Kan extension $\underline{F}_K, \underline{\psi})$ of F along any functor $K : \mathbb{X} \rightarrow \mathbb{Y}$ by replacing the slice construction by the coslice construction, and colimits by limits in the above procedure.   □

Corollary 2.4. If $(\underline{F}_K, \underline{\psi})$ is the right Kan extension of a functor F along any **full and faithful** functor K, then the natural transformation $\underline{\psi}$ is an isomorphism.

Similarly, if $(\overline{F}_K, \overline{\psi})$ is the left Kan extension of a functor F along any **full and faithful** functor K, then the natural transformation $\overline{\psi}$ is an isomorphism.

Proof. Note that for all $X \in \mathbb{X}$, $K\underline{F}(X) = \underline{F}(K(X)) := \lim(\pi_{K(X)\downarrow K})$

Since K is full and faithful, every $K(f) : K(X) \rightarrow K(X') \in \mathbb{Y}$ corresponds to a unique $f : X \rightarrow X' \in \mathbb{X}$. Then for all $X \in \mathbb{X}$, $(KX, 1_{KX})$ is an initial object in the coslice category $(KX \downarrow K)$. Thereby, the diagram $\pi_{KX\downarrow K}$ contains all the maps radiating from X. Hence, $\lim(\pi_{KX\downarrow K}F) = F(X)$, thereby, $\underline{\psi}$ is an isomorphism.

The argument for the left Kan extension is dual to the above proof.   □

We use this procedure to compute extensions of resource monotones which are functors into a posetal category (a poset considered as a category), see Section 3.4.

3 Kan extensions of Resource Measures

3.1 Resource Theories as partitioned Categories

We introduce partitioned Categories as a framework for resource theories, and functors for partitioned Categories to describe relationships between resource theories.

Definition 3.1. A **partitioned category (pCat)** $(\mathbb{X}, \mathbb{X}_f)$ consists of a category $\mathbb{X}$ and a chosen subcategory $\mathbb{X}_f$ of free transformations with the inclusion being bijective on objects.

The objects of the category are interpreted as **resources** and the maps to be **resource transformations**. The subcategory includes all objects and those transformations which are designated to be **free**.

The following are a few examples of resource theories as pCats:

Randomness

Cryptographic protocols use randomness as an essential resource for establishing secure communication of devices by generating random keys. The degree of randomness determines how secure the communication channel is. Randomness is also used in computer algorithms to solve certain problems. In other words, randomness is an essential computational resource of practical use. Entropy is used as measure of randomness: in particular, Shannon entropy quantifies randomness in that it expresses the average surprisal on the outcome of a random experiment. Entropy has been studied in the context of randomness using the category FinProb (renamed below as Detmn) [1] and [10, Example 2.5]. The following is a resource theory of randomness:

Example 3.2. $(\mathsf{Rand}, \mathsf{Detmn})$:
(Detmn is the chosen sub-category of free transformations in Rand)

Resources: (X, p) where X is a finite set and p is a probability distribution over X.

X can be interpreted of as a set of possible states of a system and p be the probability distribution over the states.

Resource Transformations: $M : (X, p) \to (Y, q)$ is a real $|X| \times |Y|$ row stochastic matrix (rows sum to 1) such that $pM = q$.

A resource transformation $M : (X, p) \to (Y, q)$ is row stochastic if and only if for all $x \in X$, M_x is a probability distribution: suppose the system is in state x, then the stochastic process produces states $y \in Y$ with probability M_{xy}. The requirement that $pM = q$ means that under the stochastic process M, the probability of Y being in state y after process M on X is given by $\sum_{x \in X} M_{xy} p_x$.

Identity transformations: Identity matrices

Composition: Suppose $(X, p) \xrightarrow{M} (Y, q) \xrightarrow{N} (Z, s)$, then $(X, p) \xrightarrow{MN} (Z, s)$ is defined as the matrix multiplication

Free transformations: A resource transformation $(X, p) \xrightarrow{M} (Y, q)$ is free if it is deterministic, that is, M is simply a function $X \to Y$. Hence, for all $x \in X, y \in Y$, $M_{xy} \in \{0, 1\}$

Non-uniformity

Pure states represent states on which the experimenter has maximum information. These conditions are often very hard to achieve in concrete settings due to the presence of external noise. In such cases, the

state is called mixed, and can be expressed as a convex combination of pure states. From this perspective, it is clear that pure states represent the maximal resource and the closer a state is to a pure state, the more resourceful it is. Therefore, the least resourceful state of any system is the *maximally mixed state*, which can be expressed as a uniform probability distribution over the states of the system. [14]

Example 3.3. $(\mathsf{Rand}, \mathsf{Uniform})$:

Resources, transformations, identity and composition: Same as example 3.2

Free transformation: A map $(X, p) \xrightarrow{U} (Y, q)$ is free if U is a uniform matrix. A row stochastic matrix $(X, p) \xrightarrow{M} (Y, q)$ is uniform if for all $y \in Y$,

$$\sum_{x \in X} M_{xy} = 1$$

The columns of M sum to $|X|/|Y|$. When U is a square matrix, it is doubly stochastic.

Note that, a uniform probability distribution $u := (1/n, 1/n, 1/n, \cdots, 1/n)$ is simply the uniform matrix $(\{*\}, (1)) \to (X, u)$, which is u itself.

$(\mathsf{Rand}, \mathsf{Uniform})$ consists of classical probabilistic states. A non-uniformity theory based on quantum states is as follows:

Example 3.4. $(\mathsf{qRand}, \mathsf{qUniform})$

Resources: (ρ, H) where $\rho : H \to H \in L(H)$ is a quantum state, also known as density matrix (a positive semi-definite operator with trace 1), and H is a finite-dimensional Hilbert space.

Resource transformations: $(\rho, H) \xrightarrow{\mathscr{E}} (\sigma, K)$ is a quantum channel $\mathscr{E} : L(H) \to L(K)$ such that $\mathscr{E}(\rho) = \sigma$

Composition and Identity transformations: Usual composition of quantum channels and identity channels

Free transformations: Unital quantum channels i.e., $\mathscr{E} : L(H) \to L(K)$ such that $\mathscr{E}\left(\frac{1}{\dim(H)} \mathbb{1}_H\right) = \frac{1}{\dim(K)} \mathbb{1}_K$, where $\mathbb{1}_H \in L(H)$ is the identity matrix. In other words, unital channels preserve maximally mixed states.

Entanglement

Entanglement is one of the most important quantum resources, and it is used in several communication scenarios, such as quantum teleportation [3] or dense coding [4]. It is known that local operations and classical communication (LOCC) cannot increase the entanglement of a quantum state [19]. Hence, when entanglement is considered to be a resource, LOCC operations are precisely the free transformations of this resource theory. The basic setting in which entanglement is studied involves quantum states over two systems, which is referred to as "bipartite entanglement".

A resource theory of bipartite entanglement is constructed as follows. The following resource theory is obtained by applying the coslice (state) construction on [10, Example 3.7]:

Example 3.5. $(\mathsf{Bip}, \mathsf{LOCC})$:

Resources: $\rho \in L(H \otimes K)$ is a quantum state which is a positive semi-definite operator with trace 1, and H, K are finite-dimensional Hilbert spaces.

Resource transformations: $\rho \xrightarrow{\mathscr{E}} \sigma$ is a quantum channel (completely positive trace preserving map) such that $\mathscr{E}(\rho) = \sigma$

Free transformations: Local operations and classical communication

The composition is the usual composition of identity channels.

Distinguishability

In some situations it is important to consider pairs of quantum states and evaluate how different they are from each other. To this end, various quantifiers have been defined, such as the trace distance, the fidelity [37] or quantum divergences [16, 13]. These quantifiers all show that, whenever the same channel is applied to each element of a pair of quantum states, in general our ability to distinguish the resulting states is decreased. This suggests setting up a resource theory of the distinguishability, also known as quantum relative majorization [28, 7].

A resource theory of quantum distinguishability is given as follows [16, 13].

Example 3.6. (Distinguish, Processing):

Resources : $((\rho, \sigma), H)$ are pairs of quantum states, that is, $\rho, \sigma \in L(H)$ where H is a finite-dimensional Hilbert space.

Resource transformations : $(\mathscr{E}_1, \mathscr{E}_2) : ((\rho_H, \sigma_H), H) \rightarrow ((\rho_K, \sigma_K), K)$ are pairs of quantum channels $\mathscr{E}_1, \mathscr{E}_2 : L(H) \rightarrow L(K)$ such that $\mathscr{E}_1(\rho_H) = \rho_K$ and $\mathscr{E}_2(\sigma_H) = \sigma_K$

Composition and identity transformations : $(\mathscr{E}_1, \mathscr{E}_2)(\mathscr{E}_3, \mathscr{E}_4) := (\mathscr{E}_1\mathscr{E}_3, \mathscr{E}_2\mathscr{E}_4)$ and identity transformations are given by pairs of identity channels

Free transformations: $(\mathscr{E}_1, \mathscr{E}_2)$ such that $\mathscr{E}_1 = \mathscr{E}_2$

3.2 Relationships between Resource Theories as pCat functors

Now that we have formalized resource theories as pCats, we can formalize the relationship between resource theories as functors of pCats. For example, classical theories of the corresponding quantum resource theories. Physical theories defined on pure states are considered as subtheories of corresponding mixed state theories. Such relationships can be formalized as functors of pCats.

Definition 3.7. A **functor of partitioned categories (pCat)**, $F : (\mathbb{X}, \mathbb{X}_f) \rightarrow (\mathbb{Y}, \mathbb{Y}_f)$, is a functor $F : \mathbb{X} \rightarrow \mathbb{Y}$ such that if $f \in \mathbb{X}_f$ then $F(f) \in \mathbb{Y}_f$ i.e., the functor preserves free transformations.

$F : (\mathbb{X}, \mathbb{X}_f)$ being a functor means that it preserves the identity transformations: $F(1_A) = 1_{F(A)}$, and it preserves the composition in $\mathbb{X}$: $F(fg) = F(f)F(g)$.

Figure 4 is a schematic of a pCat functor. The triangles represent non-free transformations, and the hollow circles represent free transformations. As one can see, a pCat functor may or may not preserve a non-free transformation.
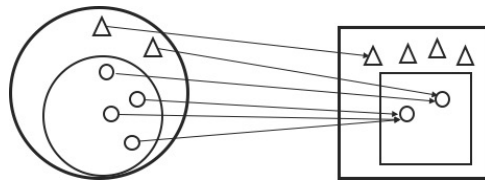

Figure 4: Schematic for functor of pCats

Definition 3.8. A pCat functor $F : (\mathbb{X}, \mathbb{X}_f) \rightarrow (\mathbb{Y}, \mathbb{Y}_f)$ is **full** if $F : \mathbb{X} \rightarrow \mathbb{Y}$ is full, and F is **faithful** if $F : \mathbb{X} \rightarrow \mathbb{Y}$ is faithful.

Let us look at a few examples of full and faithful pCat functors. For a quantum system, pure states are considered to be a subset of mixed states since mixed states are convex combination of pure states. The resource theory of pure states bipartite entanglement embeds in the general theory of bipartitle entanglement through an inclusion functor, see Example 3.9. Lemma 3.10 proves that this inclusion is a pCat functor.

Example 3.9. The resource theory of bipartite pure-state entanglement, $(\mathsf{PureBip}, \mathsf{LOCC_p})$ has pure quantum states ρ, where $\rho \in L(H \otimes K)$, as resources. A resource transformation is a quantum channel $\mathscr{E} : \rho \to \sigma$ such that $\mathscr{E}(\rho) = \sigma$ where $\sigma \in L(H' \otimes K')$. The free transformations are LOCC operations between pure bipartite states, here denoted as $\mathsf{LOCC_p}$.

Lemma 3.10. *The inclusion* $i : \mathsf{PureBip} \hookrightarrow \mathsf{Bip}$ *defined to be identity on objects and maps is a full and faithful pCat functor* $i : (\mathsf{PureBip}, \mathsf{LOCC_p}) \hookrightarrow (\mathsf{Bip}, \mathsf{LOCC})$.

Proof. The inclusion is a pCat functor since $\mathsf{LOCC_p} \hookrightarrow \mathsf{LOCC}$. Moreover, $i : \mathsf{PureBip} \hookrightarrow \mathsf{Bip}$ is full and faithful inclusion.   □

Classical theories are considered as sub-theories of quantum theories. This gives an inclusion functor classical distinguishability into quantum distinguishability. The following is the resource theory for classical distinguishability and is referred to as classical relative majorization in [30, 31, 32, 28, 7]:

Example 3.11. In the resource theory of classical distinguishability, $(\mathsf{cDistinguish}, \mathsf{cProcessing})$, a resource $((p,q), X)$ is a pair of probability distributions $p := (p_1, \cdots, p_{|X|})$ and $q := (q_1, \cdots, q_{|X|})$ over a finite set X. Resource transformations $(M, M') : ((p,q), X) \to ((p', q'), Y)$ where M and M' are pairs of row stochastic matrices such that $pM = p'$ and $qM' = q'$. Free transformations are (M, M') such that $M = M'$.

Example 3.12. The inclusion $i : \mathsf{cDistinguish} \hookrightarrow \mathsf{Distinguish}$ is defined as follows:

- For all resources $((p,q), X) \in \mathsf{cDistinguish}$, $i((p,q), X) := ((\rho^p, \rho^q), \mathbb{C}^{|X|})$ where $[\rho^p]_{ij} = \delta_{ij} p_i$, $1 \le i \le |X|, 1 \le j \le |X|$. ρ^p and ρ^q are diagonal density matrices with the probability distributions p and q as their diagonals respectively.

- Given a transformation $(M, M') : ((p,q), X) \to ((p', q'), Y)$, $i((M, M')) := (\mathscr{E}, \mathscr{E}')$ where $\mathscr{E}$ and $\mathscr{E}'$ are determined by M and M' respectively as follows.

 For any quantum state (positive semi-definite operator of trace 1 on a Hilbert Space H),

$$\mathscr{E}(\rho) = \sum_{i,j} B_{ij} \rho B_{ij}^\dagger \qquad (3.1)$$

  where $B_{ij} = \sqrt{M_{ij}} |j\rangle \langle i|$ where $1 \le i, j \le |X|$ and $B_{ij}^\dagger$ is its adjoint (cf. [37]). ($|j\rangle$ is a column vector with 1 at position j and zero elsewhere.)

Lemma 3.13. *The inclusion* $i : \mathsf{cDistinguish} \hookrightarrow \mathsf{Distinguish}$ *defined in Example 3.12 is full and faithful (or fully faithful) pCat functor* $i : (\mathsf{cDistinguish}, \mathsf{cProcessing}) \hookrightarrow (\mathsf{Distinguish}, \mathsf{Processing})$.

Example 3.14. Closely, related to Example 3.12, is the inclusion of Rand into qRand. The inclusion $i : \mathsf{Rand} \hookrightarrow \mathsf{qRand}$ is defined as follows: for all $(p, X) \in \mathsf{Rand}$, $i((p, X)) := (\rho^p, \mathbb{C}^{|X|})$, and for row stochastic matrices $M \in \mathsf{Rand}$, $i(M)$ is defined as in eqn (3.1).

Lemma 3.15. *The inclusion* $i : \mathsf{Rand} \hookrightarrow \mathsf{qRand}$ *defined in Example 3.14 is a full and faithful pCat functor* $i : (\mathsf{Rand}, \mathsf{Uniform}) \hookrightarrow (\mathsf{qRand}, \mathsf{qUniform})$.
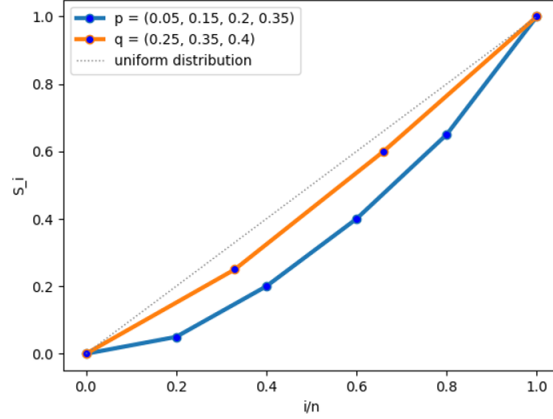
Figure 5: The Lorenz curve of q is majorized by the Lorenz curve of p

3.3 Preorder collapse and monotones

One of the major goals of resource theories is to the identify necessary and sufficient conditions for the existence of a free transformation between two resources. Once such conditions are identified, one can choose to 'forget' the different possible ways in which resource A can be converted to resource B freely, and only 'remember' if there exists a free transformation from A to B.

The necessary and sufficient conditions for the existence of a free transformation between pairs of resources define an equivalence class on the resource theory (freely inter-convertible resources are considered to be equivalent) and a preorder on the equivalence classes. Such necessary and sufficient conditions can be encoded into a pCat functor from the resource theory. On applying this functor, the resource theory collapses into a preorder:

Definition 3.16. Given a resource theory $(\mathbb{X}, \mathbb{X}_f)$ and a preorder $(\mathrm{ob}(\mathbb{X}), \mathrm{order})$ where $\mathrm{ob}(\mathbb{X})$ refers to the set of objects of $\mathbb{X}$, a **preorder collapse** of the resource theory $(\mathbb{X}, \mathbb{X}_f)$ is a pCat functor $(\mathbb{X}, \mathbb{X}_f)$ $\to (\mathrm{chaos}_\mathbb{X}, \mathrm{order}_\mathbb{X})$, where $\mathrm{chaos}_\mathbb{X}$ is the indiscrete (chaotic) category with the same objects as $\mathbb{X}$, and for any two objects $A, B \in \mathrm{chaos}_\mathbb{X}$, the transformation $A \to B \in \mathrm{order}_\mathbb{X}$ if "A order B" is true.

Let us look at an example of a preorder collapse of $(\mathrm{Rand}^{\mathrm{op}}, \mathrm{Uniform})$ determined by the majorization relation [26]. Suppose $p := (p_1, p_2, p_3, \cdots, p_n)^\uparrow$ and $q := (q_1, q_2, q_3, \cdots, q_m)^\uparrow$ such that the elements of the distribution are in increasing order. We say q is majorized by p written as $q \preceq p$ if the Lorenz curve [26, 23] of p lies either completely below the Lorenz curve of q (see Figure 5) or coincides with it. This means that q is more uniform than p.

Lorenz curve [23, 26, 14] $L(p)$ for a probability distribution $p := (p_1, p_2, \cdots, p_n)$ is characterized as the linear interpolation of points $(i/n, \sum_{k=1}^i p_k)$, where $i = 0, 1, \cdots, n$; see Figure 5.

Example 3.17. Define $P : \mathrm{Rand} \to \mathrm{chaos}_{\mathrm{Rand}}$ as follows: for each probability distribution $p \in \mathrm{Rand}$, $P(p) := p$; each $M : p \to q$ is mapped to the unique arrow $p \to q$.

The functors into chaotic categories are determined by the objects. It is straightforward that P as defined above is a functor. The following theorem establishes that $P : (\mathrm{Rand}^{\mathrm{op}}, \mathrm{Uniform}) \to (\mathrm{chaos}_{\mathrm{Rand}}, \preceq_{\mathrm{Rand}})$ is pCat functor.

Theorem 3.18. *[17] Given two finite probability distributions p and q, $q \preceq p$ if and only if there exists a uniform matrix $U : p \to q$ such that $pU = q$.*

Corollary 3.19. The map $P : (\mathrm{Rand}^{\mathrm{op}}, \mathrm{Uniform}) \to (\mathrm{chaos}_{\mathrm{Rand}}, \preceq)$ is a preorder collapse.

Proof. By Theorem 3.18, if $U : p \rightarrow q \in$ Uniform, then $P(q) \preceq P(p)$.   □

Given a resource theory, one can assign a real value to each resource such that the assignment respects the preorder defined by the theory. In this respect, a monotone is a function $f : R \rightarrow [0, \infty]$ where R is a set of resources which preserves the preorder on the equivalence class of resources. To represent a monotone as a pCat functor, the poset $([0, \infty], \leq)$ is defined as a pCat as follows:-

Definition 3.20. The poset $([0, \infty], \leq)$ is encoded as the pCat $(\text{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ where $\text{chaos}_{[0,\infty]}$ is the chaotic category with objects as $r \in [0, \infty]$ and the free transformations are those maps respecting the $\leq$ order ($m \rightarrow n \in \leq_{[0,\infty]}$ if and only if $m \leq n$).

Definition 3.21. A **monotone** for a resource theory $(\mathbb{X}, \mathbb{X}_f)$ is a pCat functor $F : (\mathbb{X}, \mathbb{X}_f) \rightarrow (\text{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$.
An **op-monotone** is a contravariant monotone, that is,

$$F : (\mathbb{X}, \mathbb{X}_f) \rightarrow (\text{chaos}_{[0,\infty]}, \leq_{[0,\infty]})^{\text{op}} := (\text{chaos}_{[0,\infty]}^{\text{op}}, \geq_{[0,\infty]}).$$

Even though op-monotones are more frequently used in resource theories, we defined the codomain of a monotone to be $(\text{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ because, in general, an arrow $a \rightarrow b$ in a posetal category refers to $a \leq b$, and such ordering becomes relevant when one computes inf and sup of a subset in the poset, see Section 3.4. Let us look at a few examples of monotones:

In information theory, Shannon entropy is a well-known measure of randomness or uncertainty in the outcome when a random experiment (experiment with multiple outcomes) is repeated one or more times. The value of Shannon entropy lies in [0,1] where 0 represents absolute certainty and 1 represents maximum uncertainty. In the following example, we construct a monotone for the resource theory of randomness, $(\text{Rand}, \text{Detmn})$, and an op-monotone $(\text{Rand}, \text{Uniform})$ based on the Shannon entropy:

Example 3.22. Define Shannon : Rand $\rightarrow \text{chaos}_{[0,\infty]}$ as follows:

- For all finite probability distributions, $(X, p) \in$ Rand, $\text{Shannon}(p) := H(p)$ where $H(p)$ is the Shannon entropy of p:
$$H(p) := - \sum_{1 \leq i \leq |X|} p_i \log p_i$$

- For all $(X, p) \xrightarrow{\text{Shannon}} (X, q) \in$ Rand, then $F(M)$ is the unique arrow $H(p) \rightarrow H(q)$

It is straightforward that Shannon : Rand $\rightarrow \text{chaos}_{[0,\infty]}$ is a functor. We note that the functor Shannon acts as a monotone for $(\text{Rand}, \text{Detmn})$ and as an op-monotone for $(\text{Rand}, \text{Uniform})$.

Lemma 3.23. *[11] Suppose* Shannon : $(X, p) \rightarrow (Y, q) \in$ Detmn, *then $H(p) \geq H(q)$.*

Corollary 3.24. The map Shannon : $(\text{Rand}, \text{Detmn}) \rightarrow ([0, \infty], \geq)$ defined as in Example 3.22 is an op-monotone.

Lemma 3.25. *[14, 26] Suppose* Shannon : $(X, p) \rightarrow (Y, q) \in$ Uniform, *then $H(p) \leq H(q)$.*

Corollary 3.26. The map Shannon : $(\text{Rand}, \text{Uniform}) \rightarrow ([0, \infty], \leq)$ defined as in Example 3.22 is a monotone.

Next we describe a monotone for the resource theory, $(\text{cDistinguish}, \text{cProcessing})$:

Example 3.27. [16, 13, Definition 2] A **classical divergence** $D :$ cDistinguish $\rightarrow \text{chaos}_{[0,\infty]}$ is any functor, that for any resource $((p, q), X) \in$ cDistinguish and a resource transformation $(M, M) \in$ cDistinguish, satisfies the data processing inequality:

$$D(p, q) \geq D(pM, qM)$$

Lemma 3.28. *Any classical divergence* D : (cDistinguish, cProcessing $\rightarrow$ (chaos$_{[0,\infty]}$, $\geq_{[0,\infty]}$) *is an op-monotone.*

The following is a monotone for PureBip as follows [27, 37]:

Example 3.29. Define Schmidt : PureBip $\rightarrow$ chaos$_{[0,\infty]}$ to be the following: for all resources $\rho^{HK} \in$ PureBip, where $\rho^{HK} \in L(H \otimes K)$ is a pure quantum state,

$$N\left(\rho^{HK}\right) := \text{Rank}\left(\rho^{H}\right) \text{ where, } \rho^{H} := \text{Tr}_{K}\left(\rho^{HK}\right)$$

Lemma 3.30. *[27, 37]* N : (PureBip, LOCC$_p$) $\rightarrow$ (chaos$_{[0,\infty]}$, $\geq_{[0,\infty]}$) *is an op-monotone.*

3.4 Kan Extensions of monotones

Now, we set up resource theories to apply Kan extensions for extending monotones from one resource theory to another when there exists a pCat functor between them.

Given a monotone M : $(\mathbb{X}, \mathbb{X}_f) \rightarrow$ (chaos$_{[0,\infty]}$, $\leq_{[0,\infty]}$) and a pCat functor K : $(\mathbb{X}, \mathbb{X}_f) \rightarrow (\mathbb{Y}, \mathbb{Y}_f)$, one could desire to extend M to obtain monotones on $(\mathbb{Y}, \mathbb{Y}_f)$. Observe that a monotone M : $(\mathbb{X}, \mathbb{X}_f)$ $\rightarrow$ (chaos$_{[0,\infty]}$, $\leq_{[0,\infty]}$) is concerned only with the free transformations: if $f : A \rightarrow B \in \mathbb{X}_f$, then $M(A) \leq M(B)$; otherwise $M(A) \rightarrow M(B)$ is the unique arrow signifying that there is no order between A and B. Since $\mathbb{X}_f$ includes all the objects of $\mathbb{X}$, and the monotone M is concerned with only free transformations, in order to extend M : $(\mathbb{X}, \mathbb{X}_f) \rightarrow$ (chaos$_{[0,\infty]}$, $\leq_{[0,\infty]}$) it suffices to extend M_f : $\mathbb{X}_f \rightarrow \leq_{[0,\infty]}$ which is defined as follows :-

$$M_f : \mathbb{X}_f \rightarrow \leq_{[0,\infty]}; \quad A \xrightarrow{f} B \mapsto M(A) \leq M(B)$$

Definition 3.31. Let $(\mathbb{X}, \mathbb{X}_f) \xrightarrow{K} (\mathbb{Y}, \mathbb{Y}_f)$ be a pCat functor. Let M : $(\mathbb{X}, \mathbb{X}_f) \rightarrow$ (chaos$_{[0,\infty]}$, $\leq_{[0,\infty]}$) be a monotone for the resource theory $(\mathbb{X}, \mathbb{X}_f)$.

(a) The **minimal extension**[1] $\underline{M}_K$: $\mathbb{Y}_f \rightarrow \leq_{[0,\infty]}$ of M along K is the right Kan extension of the functor M_f : $\mathbb{X}_f \rightarrow \leq_{[0,\infty]}$ along the functor K_f : $\mathbb{X}_f \rightarrow \mathbb{Y}_f$; $K_f(h) := K(h)$ (see Figure 6-(a)).

(b) The **maximal extension** $\overline{M}_K$: $\mathbb{Y}_f \rightarrow \leq_{[0,\infty]}$ of M along K is the left Kan extension of the functor M_f : $\mathbb{X}_f \rightarrow \leq_{[0,\infty]}$ along the functor K_f : $\mathbb{X}_f \rightarrow \mathbb{Y}_f$; $K_f(h) := K(h)$ (See Figure 6-(b)).
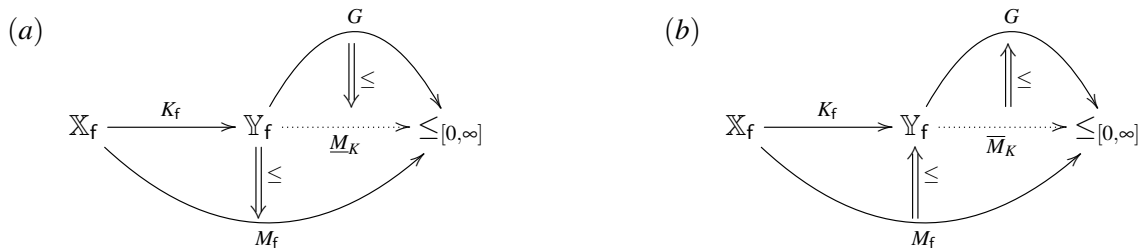

Figure 6: (a) Minimal (right Kan) extension   (b) Maximal (left Kan) extension

Let us unpack the definition of minimal and maximal extensions of a monotone in Definition 3.31. Any category given by a poset with suprema and infima is both complete and cocomplete. Since $([0,\infty], \leq)$ is such a poset, by Theorem 2.3 one can compute the minimal extension ($\underline{M}_K$) and the maximal extension ($\overline{M}_K$) using equations (2.2) and (2.1) respectively:

---

[1]We follow the naming convention in [16] for monotone extensions.

Theorem 3.32. *(a) For all $Y \in \mathbb{Y}$, the minimal extension $\underline{M}_K(Y) : \mathbb{Y}_f \rightarrow \leq_{[0,\infty]}$ is given as:*

$$\underline{M}_K(Y) := \lim(\pi_{Y \downarrow K} M_f) = \inf\{M(X) \mid Y \rightarrow K(X) \in \mathbb{Y}_f\} \tag{3.2}$$

(b)For all $Y \in \mathbb{Y}$, the maximal extension $\overline{M}_K(Y) : \mathbb{Y}_f \rightarrow \leq_{[0,\infty]}$ is given as:

$$\overline{M}_K(Y) := \mathrm{colim}(\pi_{K \downarrow Y} M_f) = \sup\{M(X) \mid K(X) \rightarrow Y \in \mathbb{Y}_f\} \tag{3.3}$$

See Figure 7-(a) for a schematic of the minimal and maximal extensions of a monotone.

Usually for resource theories the codomain of the monotones is $([0,\infty]^{\mathrm{op}}, \leq_{[0,\infty]}^{\mathrm{op}}) = ([0,\infty]^{\mathrm{op}}, \geq_{[0,\infty]})$. In the computation of extensions of op-monotones, inf is flipped to sup in equation (3.2), and sup to be flipped to inf in equation (3.3):

Corollary 3.33. Suppose $M : (\mathbb{X}, \mathbb{X}_f) \rightarrow (\mathrm{chaos}_{[0,\infty]}, \geq_{[0,\infty]})$ be a monotone and $K : (\mathbb{X}, \mathbb{X}_f) \rightarrow (\mathbb{Y}, \mathbb{Y}_f)$. Then,

(a) For all $Y \in Y$, the minimal extension $\underline{M}_K(Y) : \mathbb{Y}_f \rightarrow \geq_{[0,\infty]}$ is given as:

$$\underline{M}_K(Y) := \mathrm{colim}(\pi_{Y \downarrow K} M_f) = \sup\{M(X) \mid Y \rightarrow K(X) \in \mathbb{Y}_f\} \tag{3.4}$$

(b)For all $Y \in Y$, the maximal extension $\overline{M}_K(Y) : \mathbb{Y}_f \rightarrow \geq_{[0,\infty]}$ is given as:

$$\overline{M}_K(Y) := \lim(\pi_{K \downarrow Y} M_f) = \inf\{M(X) \mid K(X) \rightarrow Y \in \mathbb{Y}_f\} \tag{3.5}$$

Proof. Note that $(\mathrm{chaos}_{[0,\infty]}, \geq_{[0,\infty]}) = (\mathrm{chaos}_{[0,\infty]}^{\mathrm{op}}, \leq_{[0,\infty]}^{\mathrm{op}}) =: (\mathrm{chaos}_{[0,\infty]}, \leq_{[0,\infty]})^{\mathrm{op}}$. Limits in $(\mathrm{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ are colimits in $(\mathrm{chaos}_{[0,\infty]}, \leq_{[0,\infty]})^{\mathrm{op}}$. $\square$

Figure 7 - (b) and (c) visualizes the difference in computation of minimal extension of a (regular) monotone and an op-monotone.

Note that equation (3.4) is same as [16, Equation 2] and equation (3.5) is same as [16, Equation 3]. Let us have a closer look at equations (3.4) and (3.5). The minimum extension $\underline{M}_K$ assigns to any resource $Y \in \mathbb{Y}_f$ the value of a resource $X \in \mathbb{X}_f$ such that the value of X is lowest among the value of all those resources which can be transformed freely to Y under K ($KX \rightarrow Y$). If there exists no such $X \in \mathbb{X}_f$ such that KX can be transformed to Y using a free transformation, then $\underline{M}_K(Y) = 0$ (colimit of the empty diagram is the initial object).

Similarly, the maximal extension $\overline{M}_K$ assigns to any resource $Y \in \mathbb{Y}_f$ the value of a resource $X' \in \mathbb{X}_f$ such that the value of X' is the highest among the value of all those resources which Y can be transformed to freely under K ($Y \rightarrow KX$). If there does not exist any such $X \in \mathbb{X}_f$ which Y can be transformed to using a free transformation, then $\underline{M}_K(Y) = \infty$ (limit of the empty diagram is the terminal object).

We define what it means for the computed extensions to be optimal:

Definition 3.34. The minimal extension of a monotone is **optimal** if for any other monotone $G : (\mathbb{Y}, \mathbb{Y}_f) \rightarrow (\mathrm{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ such that for all $X \in \mathbb{X}$, $G(K(X)) \leq M(X)$, we have that

$$G(Y) \leq \underline{M}_K(Y)$$

The maximal extension of a monotone is **optimal** if for any other monotone $G : (\mathbb{Y}, \mathbb{Y}_f) \rightarrow (\mathrm{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ such that for all $X \in \mathbb{X}$, $M(X) \leq G(K(X))$, we have that
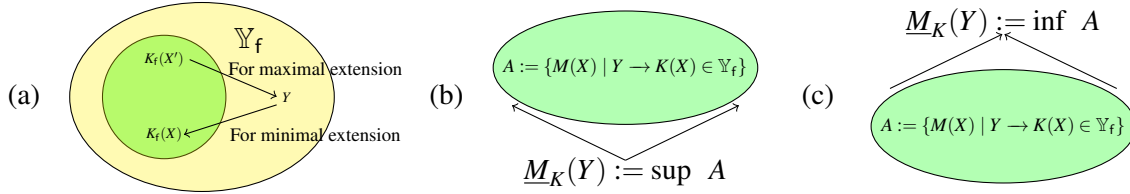
$$\overline{M}_K(Y) \leq G(Y)$$

Figure 7: (a) Schematic of minimal and maximal extensions of any monotone along K; (b) Minimal extension of monotone M; (c) Minimal extension of an op-monotone M

For the extensions of an op-monotone to be optimal, "$\leq$" is replaced by "$\geq$" in definition 3.34.

Theorem 3.35. *Let $\underline{M}_K$ and $\overline{M}_K$ be minimal and maximal extensions of a monotone $M : (\mathbb{X}, \mathbb{X}_f) \to \left(\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]}\right)$ along a pCat functor $K : (\mathbb{X}, \mathbb{X}_f) \to (\mathbb{Y}, \mathbb{Y}_f)$ as per definition 3.31. Then,*

(a) **Reduction:** *For all $X \in \mathbb{X}$,*
$$\underline{M}_K(K_f(X)) \leq M(X) \leq \overline{M}_K(K_f(X))$$

(b) **Monotonicity:** *For all $f : A \to B \in \mathbb{Y}_f$,*

$$\underline{M}_K(A) \leq \underline{M}_K(B) \text{ and } \overline{M}_K(A) \leq \overline{M}_K(B)$$

(c) **Optimality:** *$\underline{M}_K$ and $\overline{M}_K$ are optimal.*

Proof.

(a) Since $(\underline{M}_K, \leq)$ is the right Kan extension, for all $X \in \mathbb{X}$, $\underline{M}_K(K_f(X)) \leq M_f(X) = M(X)$ (see Figure 6-(a)).

Since $(\overline{M}_K, \leq)$ is the left Kan extension, for all $X \in \mathbb{X}$, $M(X) = M_f(X) \leq \overline{M}_K(K_f(X))$ (see Figure 6-(b)).

(b) Monotonicity follows from functoriality of $\underline{M}_K$ and $\overline{M}_K$

(c) The extensions are optimal by construction (see Figure 6 for the universal properties).

□

In the above lemma, Statement (a) tells us that the minimal and maximal extensions are respectively a lower and upper bound for M on $\mathbb{X}$. Statement (b) assures that the extensions are monotonic on free transformations. Statement (c) assures that the minimal and maximal extensions are respectively the greatest lower bound and the least upper bound for any other extension of M along K, hence are optimal.

Corollary 3.36. *Let $\underline{M}_K$ and $\overline{M}_K$ be minimal and maximal extensions of an op-monotone $M : (\mathbb{X}, \mathbb{X}_f) \to \left(\mathsf{chaos}_{[0,\infty]}, \geq_{[0,\infty]}\right)$ along a pCat functor $K : (\mathbb{X}, \mathbb{X}_f) \to (\mathbb{Y}, \mathbb{Y}_f)$ as per definition 3.31. Then, the following properties hold for the extensions:*

(a) **Reduction:** *For all $X \in \mathbb{X}$,*
$$\underline{M}_K(K(X)) \geq M(X) \geq \overline{M}_K(K(X))$$

(b) **Monotonicity:** *For all $f : A \to B \in \mathbb{Y}_f$,*

$$\underline{M}_K(A) \geq \underline{M}_K(B) \text{ and } \overline{M}_K(A) \geq \overline{M}_K(B)$$

(c) **Optimality:** $\underline{M}_K$ and $\overline{M}_K$ are optimal.

Proof. $\left(\mathsf{chaos}_{[0,\infty]}, \geq_{[0,\infty]}\right) = \left(\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]}\right)^{\mathsf{op}}$ $\hfill\square$

Our Corollary 3.36 corresponds to [16, Theorem 1]. However, in contrast to the proof in [16], our proof uses only the structural properties of the extensions rather than the formula used to compute them. Moreover, Lemma 3.35 is more general since in [16, Theorem 1], $K : (\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \to (\mathbb{Y}, \mathbb{Y}_{\mathsf{f}})$ is fixed to be a full and faithful inclusion.

[16, Theorem 1 - (a)] can be recovered precisely by fixing K to be full and faithful functor in Corollary 3.36:

Corollary 3.37. If $\underline{M}_F$ and $\overline{M}_F$ are minimal and maximal extensions respectively of an op-monotone $M : (\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \to \left(\mathsf{chaos}_{[0,\infty]}, \geq_{[0,\infty]}\right)$ along a full and faithful (ff) functor $F : (\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \to (\mathbb{Y}, \mathbb{Y}_{\mathsf{f}})$, then:

(a) **ff-Reduction:** The extensions exactly preserves the value of the resources in $\mathbb{X}$ under the action of F:
$$\overline{M}_F(F(X)) = M(X) = \underline{M}_F(F(X))$$

(b) **ff-Optimality:** For any other monotone $G : (\mathbb{X}, \mathbb{X}_{\mathsf{f}}) \to \left(\mathsf{chaos}_{[0,\infty]}, \geq_{[0,\infty]}\right)$, which exactly preserves the value of the resources in $\mathbb{X}$ under the action of F, that is, $(G(F(X)) = M(X))$, then for all $Y \in \mathbb{Y}$:

$$\overline{M}_F(F(Y)) \geq G(Y) \geq \underline{M}_F(F(Y))$$

Proof. Statement (a) follows from Lemma 2.4.

For Statement (b), it is given that for all $X \in \mathbb{X}$, $G(F(X)) = F(X)$. From Lemma 3.35-(c), it follows that for all $Y \in \mathbb{Y}$,
$$\underline{M}_F(Y) \geq G(Y) \geq \overline{M}_F(Y)$$
$\hfill\square$

3.4.1　Extending bipartite entanglement monotone from pure to mixed states:

Example 3.38. We introduced the op-monotone $\mathsf{Schmidt} : (\mathsf{PureBip}, \mathsf{LOCC_p}) \to \left(\mathsf{chaos}_{[0,\infty]}, \geq_{[0,\infty]}\right)$ in Example 3.29. Let us extend the monotone from pure bipartite states to mixed states along $i :$ $(\mathsf{PureBip}, \mathsf{LOCC_p}) \hookrightarrow (\mathsf{Bip}, \mathsf{LOCC})$ (defined in Lemma 3.10), something that was already done in [36], however without the general machinery for computing extensions.

Figure 8 presents the diagrams corresponding to minimal and maximal extensions of the monotone Schmidt along the inclusion. The minimal and the maximal extensions are computed using equations (3.4) and (3.5) respectively.

It was pointed out in [16] that the definition for the Schmidt entanglement monotone on mixed bipartite states introduced in [36] coincides with equation (3.5) referring to the maximal extension of Schmidt.

3.4.2　Extending classical divergences

Next we examine the properties of extensions of classical divergences to quantum setting:

Lemma 3.39. *Let* $D : (\mathsf{cDistinguish}, \mathsf{cProcessing}) \to \left(\mathsf{chaos}_{[0,\infty]}, \geq_{[0,\infty]}\right)$ *be a classical divergence as defined in Definition 3.27. Let* $\underline{D}_i$ *and* $\overline{D}_i$ *be the minimal and maximal extensions respectively of D along* $i : (\mathsf{cDistinguish}, \mathsf{cProcessing}) \hookrightarrow (\mathsf{Distinguish}, \mathsf{Processing})$. *Then the extensions satisfy the following properties:*
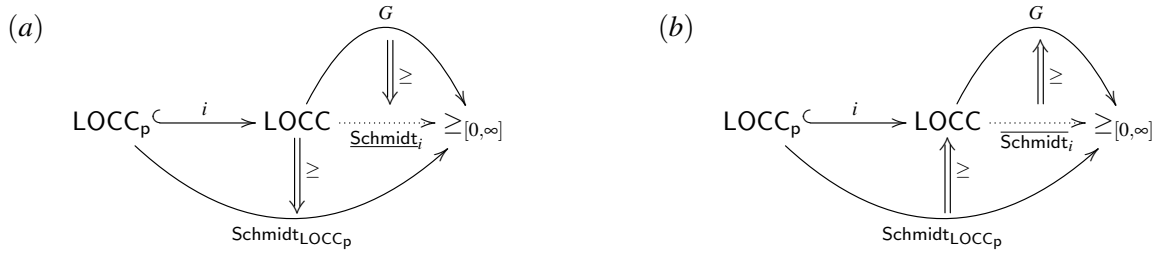
Figure 8: (a) Minimal extension of Schmidt : $(\mathsf{PureBip}, \mathsf{LOCC_p})$ along $i : (\mathsf{PureBip}, \mathsf{LOCC_p}) \hookrightarrow$ $(\mathsf{Bip}, \mathsf{LOCC})$; (b) Maximal extension of Schmidt : $(\mathsf{PureBip}, \mathsf{LOCC_p})$ along $i : (\mathsf{PureBip}, \mathsf{LOCC_p}) \hookrightarrow$ $(\mathsf{Bip}, \mathsf{LOCC})$

(a) **Reduction**: *For all* $((p,q),X) \in \mathsf{Distinguish}$, $\underline{D}_i(p||q) = D(p||q) = \overline{D}_i(p||q)$

(b) **Monotonicity**: *For any* $M : ((p,q),X) \to ((p',q'),Y)$, $\underline{D}_i(p||q) \geq \underline{D}_i(pM||qM)$ *and* $\overline{D}_i(p||q) \geq$ $\overline{D}_i(pM||qM)$

(c) **Optimality**: *Suppose* $D' : (\mathsf{Distinguish}, \mathsf{Processing}) \to \left(\mathsf{chaos}_{[0,\infty]}, \geq_{[0,\infty]}\right)$ *is any pCat functor such that for all* $((p,q),X) \in \mathsf{cDistinguish}$, $D'(i((p,q),X)) = D(((p,q),X))$. *Then for all* $((\rho,\sigma),H) \in$ $\mathsf{Distinguish}$,

$$\underline{D}_i(\rho||\sigma) \geq G(\rho||\sigma) \geq \overline{D}_i(\rho||\sigma) \tag{3.6}$$

Proof. By Lemma 3.13, the inclusion $i : \mathsf{cDistinguish} \hookrightarrow \mathsf{Distinguish}$ is full and faithful. Hence, statement (a) and Statement (c) follows directly from ff-Reduction and ff-Optimality properties respectively in Lemma 3.35. Statement (b) follows from Monotonicity property in Lemma 3.37-(b). $\square$

In the above statement, by the **Reduction** property, $\underline{D}_i$ and $\overline{D}_i$ reduces to classical divergence D on the classical states (pairs of density matrices with off diagonal elements to be zero). The **optimality** property ensures that, for any other quantum divergence that coincides with D on the classical states, must lie between the maximal and minimal extensions in the sense of Eqn. (3.6).

3.4.3   Extending Shannon entropy

Now we show that Kan extensions are related to some proposals of extending Shannon entropy from classical states to states of a general physical theory [2, 34, 21, 33, 8, 22]. Specifically, a measurement and a preparation extensions were proposed. Here, for simplicity, we will explain them in the context of quantum theory. In more detail, the measurement entropy H_{meas} of a quantum state ρ is defined as

$$H_{\mathrm{meas}}(\rho) := \inf_{F} H(q), \tag{3.7}$$

where the infimum is taken over all rank-one POVMs $F := \{F_j\}$, and q is a probability distribution with $q_j := \mathrm{tr}\, F_j \rho$. Recall that a POVM is a collection of positive semi-definite operators $\{F_j\}$ that sum to the identity. On the other hand, the preparation entropy H_{prep} is defined as

$$H_{\mathrm{prep}}(\rho) := \inf_{\Sigma_j \lambda_j \psi_j = \rho} H(\lambda), \tag{3.8}$$

where the infimum is over all convex decompositions $\sum_j \lambda_j \psi_i$ of the state ρ in terms of pure states ψ_j (recall that a quantum state ψ is pure if $\psi^2 = \psi$). In words, the measurement entropy H_{meas} is the smallest amount of randomness (as measured by Shannon entropy H) present in the probability distributions generated by rank-one POVMs on ρ. On the other hand, the preparation entropy H_{prep} is the smallest amount of randomness necessary to prepare ρ as an convex combination of pure states.

Let us consider the inclusion (Example 3.14) of resource theory of non-uniformity (given in Example 3.3) into quantum non-uniformity (given in Example 3.4). Consider extending the monotone Shannon : $(\mathsf{Rand}, \mathsf{Uniform}) \to (\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ (given in Example 3.22) along the inclusion as shown in Figure 9. By the Kan extensions formula in equations (3.2) and (3.3), the minimal and maximal extension of Shannon are given as follows:
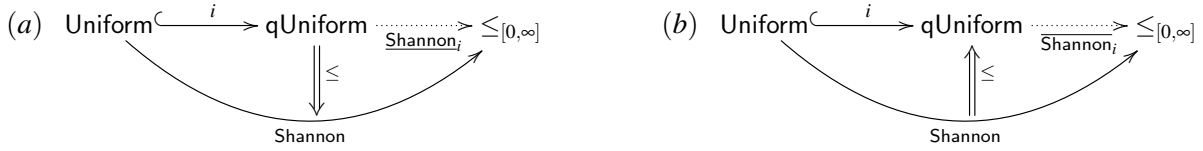

Figure 9: (a) Minimal extension of Shannon : $(\mathsf{Rand}, \mathsf{Uniform})$ along i : $(\mathsf{Rand}, \mathsf{Uniform}) \hookrightarrow (\mathsf{qRand}, \mathsf{qUniform})$; (b) Maximal extension of Shannon : $(\mathsf{Rand}, \mathsf{Uniform})$ along i : $(\mathsf{Rand}, \mathsf{Uniform}) \hookrightarrow (\mathsf{qRand}, \mathsf{qUniform})$

For all $\rho \in \mathsf{qUniform}$, the minimal extension $\underline{\mathsf{Shannon}}_i$: $\mathsf{qUniform} \to \leq_{[0,\infty]}$ is given as:

$$\underline{\mathsf{Shannon}}_i(\rho) := \inf\{\mathsf{Shannon}(p) \mid \rho \to i(p) \in \mathsf{qUniform}\} \tag{3.9}$$

For all $\rho \in \mathsf{qUniform}$, the maximal extension $\overline{\mathsf{Shannon}}_i$: $\mathsf{qUniform} \to \leq_{[0,\infty]}$ is given as:

$$\overline{\mathsf{Shannon}}_i(\rho) := \sup\{\mathsf{Shannon}(p) \mid i(p) \to \rho \in \mathsf{qUniform}\} \tag{3.10}$$

Let us have a closer look at equation (3.9) The only unital channels from a quantum to a classical system are given by rank 1 projective measurements $\{P_j\}$, where P_j are rank 1 orthogonal projectors. With this in mind, eqn. (3.9) can be rewritten as follows:

$$\underline{\mathsf{Shannon}}_i(\rho) := \inf_P H(q),$$

where the infimum is taken over all rank-one projective measurements $P := \{P_j\}$, and q is a probability distribution with $q_j := \mathrm{tr}\, P_j \rho$. Now we are going to show that $\underline{\mathsf{Shannon}}_i(\rho) = H_{\mathrm{meas}}(\rho)$. To this end, notice that $\underline{\mathsf{Shannon}}_i(\rho) \geq H_{\mathrm{meas}}(\rho)$ because the infimum in the definition of $\underline{\mathsf{Shannon}}_i(\rho)$ is over a smaller set. In theorem 5.4.15 of [33] it was shown that $H_{\mathrm{meas}}(\rho)$ is achieved by considering the spectral POVM, which is a rank-1 projective measurement. Being $\underline{\mathsf{Shannon}}_i(\rho)$ defined as the infimum over rank-1 projective measurements, then we also have $\underline{\mathsf{Shannon}}_i(\rho) \leq H_{\mathrm{meas}}(\rho)$, from which we conclude that $\underline{\mathsf{Shannon}}_i(\rho) = H_{\mathrm{meas}}(\rho)$. Since $H_{\mathrm{meas}}(\rho)$ is achieved by the spectral measurement, we know that $H_{\mathrm{meas}}(\rho) = H(p)$, where p denotes the classical vector of the spectrum of ρ. This shows that H_{meas} as defined in equation (3.7) is indeed a monotone, as it coincides with the minimal Kan extension.

Let us now have a closer look at equation (3.10). The only unital channels from a classical to a quantum system are given by preparations of a convex combination of pure states $\{\psi_j\}$ associated with

an orthonormal basis of the Hilbert Space corresponding to the quantum system, where the coefficients are the entries of the classical state on which the channel acts. With this in mind, eqn (3.10) can be rewritten as follows:

$$\overline{\mathsf{Shannon}}_i(\rho) := \sup_{\sum_j \lambda_j \psi_j = \rho} H(\lambda),$$

where the supremum is taken over all decompositions of ρ into *orthogonal* pure states. Now, we we observe that all such decompositions are diagonalizations of ρ (that is, $\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ with λ being a probability distribution), and therefore they have the same coefficients λ_j, which are the eigenvalues of ρ. In other words, $\overline{\mathsf{Shannon}}_i(\rho) = H(p)$, where p denotes the classical vector of the spectrum of ρ. Since there is only one vector λ (up to permutation) to optimize over, the supremum can be replaced with an infimum. With this in mind, we obtain an expression that is close the preparation entropy.

$$\overline{\mathsf{Shannon}}_i(\rho) := \inf_{\sum_j \lambda_j \psi_j = \rho} H(\lambda),$$

where the infimum is taken over all decompositions of ρ into *orthogonal* pure states. In Theorem 5.4.15 of [33] it was shown that $H_{\mathrm{prep}}(\rho) = H(p)$, from which we have that $\overline{\mathsf{Shannon}}_i(\rho) = H_{\mathrm{prep}}(\rho)$. This shows that H_{meas} as defined in equation (3.8) is indeed a monotone, as it coincides with the maximal Kan extension.

Notice that in this example, the minimal and maximal Kan extensions coincide.

4   Conclusion

In this article, we studied resource theories as partitioned categories (pCats) and relationship between resource theories as pCat functors thereof. A partitioned category (pCat) is a category with a chosen subcategory of free transformations. In this framework, a monotone for a resource theory can be viewed as a pCat functor from the theory into $(\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ where the pCat $(\mathsf{chaos}_{[0,\infty]}, \leq_{[0,\infty]})$ represents the partial order $([0,\infty], \leq)$.

We showed that a monotone can be extended from one theory to another using Kan extensions. We applied our framework to extend entanglement monotones for bipartite pure states to bipartite mixed states, to extend classical divergences to the quantum setting, and to extend non-uniformity monotone from classical probabilistic theory to quantum theory.

This project was inspired by Gour and Tomamichel's work [16], which uses a set-based framework to provide formulae for the minimal and maximal extensions of a monotone for a resource theory that embeds (fully and faithfully) in a larger theory. The goal of our work was to present resource theories and monotones in a framework such that the extension formulae for monotones arise naturally. We found that they are precisely given by the well-studied notion of Kan extensions. On top of providing a natural ground to study extensions of monotones, we should also note that our categorical framework is also more general than the framework in [16], in that it can be used to compute monotone extensions when the pCat functor between resource theories is not a full and faithful embedding.

Acknowledgments

References

[1] J. C. Baez, T. Fritz & T. Leinster (2011): *A characterization of entropy in terms of information loss.* Entropy 13(11), pp. 1945–1957, doi:10.3390/e13111945.

[2] H. Barnum, J. Barrett, L. Orloff Clark, M. Leifer, R. Spekkens, N. Stepanik, A. Wilce & R. Wilke (2010): *Entropy and information causality in general probabilistic theories.* New J. Phys. 12(3), p. 033024, doi:10.1088/1367-2630/12/3/033024.

[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres & W. K. Wootters (1993): *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels.* Phys. Rev. Lett. 70, pp. 1895–1899, doi:10.1103/PhysRevLett.70.1895.

[4] C. H. Bennett & S. J. Wiesner (1992): *Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states.* Phys. Rev. Lett. 69, pp. 2881–2884, doi:10.1103/PhysRevLett.69.2881.

[5] F. Borceux (1994): *Handbook of Categorical Algebra.* 1, Cambridge University Press, Cambridge, doi:10.1017/CBO9780511525858.

[6] F. G. S. L. Brandão & G. Gour (2015): *Reversible Framework for Quantum Resource Theories.* Phys. Rev. Lett. 115, p. 070503, doi:10.1103/PhysRevLett.115.070503.

[7] F. Buscemi & G. Gour (2017): *Quantum relative Lorenz curves.* Phys. Rev. A 95, p. 012110, doi:10.1103/PhysRevA.95.012110.

[8] G. Chiribella & C. M. Scandolo (2016): *Entanglement as an axiomatic foundation for statistical mechanics.* arXiv:1608.04459 [quant-ph]. Available at `http://arxiv.org/abs/1608.04459`.

[9] E. Chitambar & G. Gour (2019): *Quantum resource theories.* Rev. Mod. Phys. 91, p. 025001, doi:10.1103/RevModPhys.91.025001.

[10] B. Coecke, T. Fritz & R. W Spekkens (2016): *A mathematical theory of resources.* Inf. Comput. 250, pp. 59–86, doi:10.1016/j.ic.2016.02.008.

[11] T. M. Cover & J. A. Thomas (2006): *Elements of information theory*, 2nd edition. Wiley, Hoboken, doi:10.1002/047174882X.

[12] J. Goold, M. Huber, A. Riera, L. del Rio & P. Skrzypczyk (2016): *The role of quantum information in thermodynamics—a topical review.* J. Phys. A 49(14), p. 143001, doi:10.1088/1751-8113/49/14/143001.

[13] G. Gour (2021): *Uniqueness and Optimality of Dynamical Extensions of Divergences.* PRX Quantum 2, p. 010313, doi:10.1103/PRXQuantum.2.010313.

[14] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens & N. Yunger Halpern (2015): *The resource theory of informational nonequilibrium in thermodynamics.* Phys. Rep. 583, pp. 1–58, doi:10.1016/j.physrep.2015.04.003.

[15] G. Gour & C. M. Scandolo (2020): *Dynamical Entanglement.* Phys. Rev. Lett. 125, p. 180505, doi:10.1103/PhysRevLett.125.180505.

[16] G. Gour & M. Tomamichel (2020): *Optimal extensions of resource measures and their applications.* Phys. Rev. A 102(6), doi:10.1103/physreva.102.062401.

[17] G. H. Hardy, J. E. Littlewood & G. Pólya (1929): *Some simple inequalities satisfied by convex functions.* Messenger Math. 58(145–152), p. 310.

[18] M. Horodecki & J. Oppenheim (2013): *(Quantumness in the context of) resource theories.* Int. J. Mod. Phys. B 27(01n03), p. 1345019, doi:10.1142/S0217979213450197.

[19] R. Horodecki, P. Horodecki, M. Horodecki & K. Horodecki (2009): *Quantum entanglement.* Rev. Mod. Phys. 81, pp. 865–942, doi:10.1103/RevModPhys.81.865.

[20] D. M Kan (1958): *Adjoint functors*. Trans. Am. Math. Soc. 87(2), pp. 294–329, doi:10.1090/S0002-9947-1958-0131451-0.

[21] G. Kimura, K. Nuida & H. Imai (2010): *Distinguishability measures and entropies for general probabilistic theories*. Rep. Math. Phys. 66(2), pp. 175–206, doi:10.1016/S0034-4877(10)00025-X.

[22] M. Krumm, H. Barnum, J. Barrett & M. P. Müller (2017): *Thermodynamics and the structure of quantum theory*. New J. Phys. 19(4), p. 043025, doi:10.1088/1367-2630/aa68ef.

[23] M. O. Lorenz (1905): *Methods of Measuring the Concentration of Wealth*. Publ. Am. Stat. Assoc. 9(70), pp. 209–219, doi:10.1080/15225437.1905.10503443.

[24] M. Lostaglio (2019): *An introductory review of the resource theory approach to thermodynamics*. Rep. Prog. Phys. 82(11), p. 114001, doi:10.1088/1361-6633/ab46e5.

[25] S. Mac Lane (1998): *Categories for the working mathematician*, 2nd edition. Graduate Texts in Mathematics, Springer Science & Business Media, New York, doi:10.1007/978-1-4757-4721-8.

[26] A. W. Marshall, I. Olkin & B. C. Arnold (2011): *Inequalities: Theory of Majorization and Its Applications*. Springer Series in Statistics, Springer, New York, doi:10.1007/978-0-387-68276-1.

[27] M. A. Nielsen & I. L. Chuang (2011): *Quantum Computation and Quantum Information*, 10th anniversary edition. Cambridge University Press, Cambridge, doi:10.1017/CBO9780511976667.

[28] J. M. Renes (2016): *Relative submajorization and its use in quantum resource theories*. J. Math. Phys. 57, p. 122202, doi:10.1063/1.4972295.

[29] E. Riehl (2017): *Category Theory in Context*. Aurora: Dover Modern Math Originals, Dover Publications, Mineola, NY.

[30] E. Ruch & A. Mead (1976): *The principle of increasing mixing character and some of its consequences*. Theor. Chim. Acta 41, pp. 95–117, doi:10.1007/BF01178071.

[31] E. Ruch, R. Schranner & T. H. Seligman (1978): *The mixing distance*. J. Chem. Phys. 69(1), pp. 386–392, doi:10.1063/1.436364.

[32] E. Ruch, R. Schranner & T. H. Seligman (1980): *Generalization of a theorem by Hardy, Littlewood, and Pólya*. J. Math. Anal. Appl. 76(1), pp. 222–229, doi:10.1016/0022-247X(80)90075-X.

[33] C. M. Scandolo (2018): *Information-theoretic foundations of thermodynamics in general probabilistic theories*. Ph.D. thesis, University of Oxford. Available at https://arxiv.org/abs/1901.08054.

[34] A. J. Short & S. Wehner (2010): *Entropy in general physical theories*. New J. Phys. 12(3), p. 033023, doi:10.1088/1367-2630/12/3/033023.

[35] A. Streltsov, G. Adesso & M. B. Plenio (2017): *Colloquium: Quantum coherence as a resource*. Rev. Mod. Phys. 89, p. 041003, doi:10.1103/RevModPhys.89.041003.

[36] B. M. Terhal & P. Horodecki (2000): *Schmidt number for density matrices*. Phys. Rev. A 61, p. 040301, doi:10.1103/PhysRevA.61.040301.

[37] M. M. Wilde (2017): *Quantum Information Theory*, 2nd edition. Cambridge University Press, Cambridge, doi:10.1017/9781316809976.