Enriched structure–semantics adjunctions and monad–theory equivalences for subcategories of arities

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(Joint work with Rory Lucyshyn-Wright)

Several structure–semantics adjunctions and monad–theory equivalences have been established in category theory. Lawvere [6] developed a structure–semantics adjunction between Lawvere theories and *tractable* Set-valued functors, which was subsequently generalized by Linton [7], while Dubuc [3] established a structure–semantics adjunction between $\mathscr{V}$*-theories* and *tractable* $\mathscr{V}$-valued $\mathscr{V}$-functors for a symmetric monoidal closed category $\mathscr{V}$. It is also well known (and due to Linton) that there is an equivalence between Lawvere theories and finitary monads on Set. Generalizing this result, Lucyshyn-Wright [8] established a monad–theory equivalence for *eleutheric* systems of arities in arbitrary closed categories. Building on work of Nishizawa and Power [11], Bourke and Garner [2] subsequently proved a general monad–theory equivalence for arbitrary small subcategories of arities in locally presentable enriched categories. However, neither of the equivalences of [8] and [2] generalizes the other, and there has not yet been a general treatment of enriched structure–semantics adjunctions that specializes to those established by Lawvere, Linton, and Dubuc.

Motivated by these considerations, we develop a general axiomatic framework for studying enriched structure–semantics adjunctions and monad–theory equivalences for subcategories of arities, which generalizes many previously established results of this kind and also provides substantial new examples in topology, analysis, and differential geometry. For a subcategory of arities $\mathscr{J}$ in a $\mathscr{V}$-category $\mathscr{C}$ over a symmetric monoidal closed category $\mathscr{V}$, Linton's notion of *clone* [7] generalizes to provide enriched notions of $\mathscr{J}$*-theory* and $\mathscr{J}$*-pretheory*, which were also employed by Bourke and Garner [2]. We say that $\mathscr{J}$ is *amenable* if every $\mathscr{J}$-theory admits free algebras, and is *strongly amenable* if every $\mathscr{J}$-pretheory admits free algebras. If $\mathscr{J}$ is amenable, we obtain an idempotent structure–semantics adjunction between *admissible* $\mathscr{J}$-pretheories and $\mathscr{J}$*-tractable* $\mathscr{V}$-categories over $\mathscr{C}$, which yields an equivalence between $\mathscr{J}$-theories and $\mathscr{J}$*-nervous* $\mathscr{V}$-monads on $\mathscr{C}$. We obtain further results when $\mathscr{J}$ is strongly amenable, including the existence of algebraic colimits of $\mathscr{J}$-theories and $\mathscr{J}$-nervous $\mathscr{V}$-monads, and the strict monadicity of algebraic $\mathscr{V}$-functors. We show that many previously studied subcategories of arities are (strongly) amenable, including also the eleutheric and *bounded* subcategories of arities studied by the authors in [10], from which we recover many

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structure–semantics adjunctions and monad–theory equivalences previously established in the literature. We conclude with the result that any small subcategory of arities in a *locally bounded* closed category is strongly amenable, from which we obtain structure–semantics adjunctions and monad–theory equivalences in many convenient categories of topological and smooth spaces. In particular, as we show in [9], locally bounded closed categories include the *concrete quasitoposes* of Dubuc [4] and the convenient categories of smooth spaces of Baez and Hoffnung [1], as well as many convenient cartesian closed categories of topological spaces, including compactly generated (Hausdorff, weakly Hausdorff) spaces and the core compactly generated spaces of Escardó-Lawson-Simpson [5].

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