Unification of Modal Logic via Topological Categories

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In this paper we provide a unifying description of different types of semantics of modal logic found in the literature via the framework of topological categories. In the style of categorical logic, we establish an exact correspondence between various syntactic extensions of modal logic on one hand, including modal dependence, group agent structures, and logical dynamic, and semantic structures in topological categories on the other hand. This framework provides us a uniform treatment of interpreting these syntactic extensions in all different types of semantics of modal logic, and it deepens our conceptual understanding of the abstract structure of modal logic.

1 Introduction

Throughout the history of modal logic, many different types of semantics have been developed to interpret the modal language, with various applications in mind. Starting from the seminal work by von Wright [24] and the later extension by Hintikka in [14], the Kripkean style semantics of modal logic has been widely applied in the philosophical study of epistemology. Tarski and McKinsey in [19] have also discovered that the interior operator induced by a topological space could be used to interpret modal formulas as well, which naturally finds its connection with propositional intuitionistic logic. Other variations include neighbourhood semantics for modal logic, first suggested by Scott in [22] in order to study certain non-normal fragments of modal logic. Finally, we also have semantics of a more algebraic flavour, extending the usual algebraisation of propositional logic using Boolean algebras.

These various forms then naturally bear the following question: Is it possible to provide a *unifying* description of all types of semantic models of modal logic? To provide a positive answer, this paper starts with the following observation: In all of the above mentioned examples, in fact in many more cases, the categories of semantics of modal logic all organise themselves into *topological categories* (over **Set**).

The notion of topological categories is introduced in [1], with the aim of axiomatising the structure of those categories containing objects X equipped with certain *geometric data*, with X living in an ambient category **X**. This results in the notion of topological categories over an arbitrary base **X**. For our purpose though, we will exclusively work over **Set**, and this is our default for topological categories henceforth. The prototypical example is **Top**, the category of topological spaces, whose objects are sets equipped with a topology. We will give an overview of topological categories in Section 2, and provide another equivalent way of describing topological categories more suitable for modal logic (cf. Theorem 2.5). It can be immediately recognised that all the mentioned examples of semantics conform to such a description: Kripke models are sets equipped with a binary relation, which are often depicted diagrammatically. We've already mentioned topological spaces, and neighbourhood models are no exceptions. Perhaps surprisingly, a particular style of algebraic semantics, using *complete atomic Boolean algebra with operators* (CABAO), can also be recognised as topological or geometrical over **Set**, once we take its dual category. This is arguably an incarnation of the duality principle between algebra and geometry within

the context of modal logic. We will prove in Proposition 2.6 that all these types of semantics, and in fact much more, are instances of topological categories, hence building the foundations of unification.

But such fact alone is far from convincing that this is a good framework for unifying modal logic. The more important topic is how the semantic structures of topological categories would explain the various logical features. In this paper, we will follow the philosophy of categorical logic, establishing *exact correspondences* between different syntactic patterns of modal logic with semantic structures of topological categories. Such correspondences are witnessed by considering transformation of models, viz. functors between topological categories.

The first thing to explain is the interpretation of modalities. As we will see in more detail in Section 3, it is precisely the geometric data of a topological category that is responsible for its interpretation. Furthermore, the structure of topological categories also connects tightly with many other *extensions* of basic modal logic studied in the literature, including the multi-agency, group agency, modal dependence, logical dynamics, etc.. For each of these reasoning patterns we have established theorems (see Theorem 3.4, 4.4 and 5.3) showing that functors preserve certain structures of topological categories if, and only if, the linguistic interpretation of the corresponding fragment of modal logic remains unchanged under the transformation. These results significantly improve our conceptual understanding of modal logic, and will be the main topics of Section 4 and 5.

To the best knowledge of the author, in the current literature there has been no theoretic framework to enable all these different fragments of modal logic to be described in a uniform way for all types of semantics. Our systematic approach allows seamless generalisation of all these constructions in modal logic to any other semantics. For instance, it has been actively discussed what is the corresponding notion of common knowledge in topological semantics [4], how to extend different forms of logical dynamics to wider contexts [5], or how to develop modal dependence described in [3] and [2] for other semantic types. Our work provides a novel answer to all these different questions by accommodating them to the framework of topological categories, and it has ample potential applications.

2   Preliminaries

In many existing texts, e.g. in [1, 15], topological categories are usually introduced as *fibrations* over **Set** satisfying certain lifting properties. It is well-known from the Grothendieck construction that fibrations can be equivalently described by *indexing categories*, or functors mapping out of **Set**. For our purpose, it is this equivalent *indexing* point of view of topological categories that is more suitable for making connections with modal logic. We will discuss this in more detail below.

Recall that a *concrete category*, or a *construct*, is simply a faithful functor $U : \mathscr{A} \to$ **Set**. When it is clear from the context what the functor U is, we will simply refer to $\mathscr{A}$ as a concrete category.

Example 2.1. We take this opportunity to introduce the main examples of category of semantics:

- **Kr** denotes the category of Kripke frames, whose objects are sets equipped with a binary relation on them, with morphisms being monotone maps. It has certain useful full subcategories including **Pre** and **Eqv**, whose objects only contains preorders or equivalence relations.

- We've mentioned that **Top** will denote the category of topological spaces.

- **Nb** is the category of neighbourhood frames, whose objects are sets X equipped with a neighbourhood relation $E \subseteq X \times \wp(X)$, and whose morphisms $f : (X, E) \to (Y, F)$ are functions from X to Y satisfying a continuity condition: For any $x \in X$ and $V \subseteq Y$, $fxFV \Rightarrow xEf^{-1}V$.

- We let objects of **CABAO** be pairs (X, m) with m being an arbitrary endo-function on $\wp(X)$, and morphisms $f : (X, m) \to (Y, n)$ are functions from X to Y satisfying $f^{-1} \circ n \subseteq m \circ f^{-1}$, where we extend the order $\subseteq$ on $\wp(X)$ point-wise to the function space $\wp(X)^{\wp(Y)}$.[1]

- Besides models of the above form, to interpret modal formulas we also need evaluation functions to interpret propositional letters. For a fixed set P of propositional variables, we introduce the category **Evl** of evaluations, whose objects are pairs (X, V) with $V : \mathsf{P} \to \wp(X)$, and morphisms $f : (X, V) \to (Y, W)$ are functions from X to Y satisfying $V \subseteq f^{-1} \circ W$, where similarly the order is the point-wise extension of the subset relation on the function space $\wp(X)^{\mathsf{P}}$.

In each case, there is an evident forgetful functor to **Set** that identifies them as concrete categories. $\diamond$

Let us say a few more words on the category **CABAO**. From a well-known theorem of Tarski, we know every CABA is isomorphic to a power set algebra $\wp(X)$ (and every power set algebra is a CABA), and every morphism between them is of the form $f^{-1} : \wp(Y) \to \wp(X)$ for some function $f : X \to Y$. Hence, our definition of a CABAO as a pair (X, m) does not lose anything, and it builds in the duality, since it uses f, rather than f^{-1}, as morphisms. Notice that the morphisms we choose between CABAOs are not the *algebraic* ones, which should commute with the operators on both sides, but *lax* ones that only require an inequality. A possible intuition for this choice is to read the operators m, n as interior operators induced by a topology, and the above continuity condition is exactly saying that f is a continuous map for the two topological spaces. We will see later that such a choice makes **CABAO** topological over **Set**.

There is also an accompanying notion of *concrete functors* between concrete categories: A functor F between two concrete categories $(\mathscr{A}, |-|_{\mathscr{A}})$ and $(\mathscr{B}, |-|_{\mathscr{B}})$ is a concrete functor iff it commutes with the forgetful functors, i.e. iff it preserves the underlying sets. Obviously, each forgetful functor of $|-|$ of a construct $\mathscr{A}$ constitute a concrete functor from $(\mathscr{A}, |-|)$ to $(\mathbf{Set}, 1_{\mathbf{Set}})$, which establish **Set** as the terminal object in the (large) category of concrete categories and concrete functors.

The faithfulness of the forgetful functor of a concrete category has many consequences. For any construct $(\mathscr{A}, |-|)$, we will identify the Hom-sets $\mathscr{A}(A, B)$ simply as subsets of $\mathbf{Set}(|A|, |B|)$, and say a function $f : |A| \to |B|$ is an $\mathscr{A}$-*morphism* if it belongs to $\mathscr{A}(A, B)$. For instance, f is a **Top**-morphism if it is continuous. Faithfulness of $|-|$ also implies that each fibre $\mathscr{A}_X$ over a set X is a (possibly large) preorder — recall that a morphism in $\mathscr{A}_X$ is a morphism in $\mathscr{A}$ above id_X. If each fibre is indeed small, then we say the construct $\mathscr{A}$ is *fibre-small*. It is easy to verify that all the introduced categories in Example 2.1 have small fibres. All the constructs considered in the future will be fibre-small.

As mentioned, topological categories are constructs that satisfy certain lifting properties. For any construct $(\mathscr{A}, |-|)$, a *structured source* is defined to be a set of functions of the form $\{f_i : X \to |A_i|\}_{i \in I}$,[2] where each $A_i \in \mathscr{A}$. An *initial lift* of such a structured source is an object A in the fibre $\mathscr{A}_X$, satisfying the following universal properties: For any function $g : |B| \to |A|$, g is an $\mathscr{A}$-morphism iff $f_i \circ g : |B| \to |A_i|$ is an $\mathscr{A}$-morphism for any $i \in I$. Evidently, initial lifts are identified up to isomorphisms in the fibre $\mathscr{A}_X$.

Definition 2.2 (Topological Categories). A construct $(\mathscr{A}, |-|)$ is a *topological category* if every structured source has a *unique* initial lift.

We can break the definition of a topological category into two parts: It first requires the *existence* of initial lifts of structured sources, and it also requires the *uniqueness* of such lifts. The notion of initial lift of structured source is a generalisation of cartesian lifts for Grothendieck fibrations. In fact, cartesian lift is exactly initial lift for a *singleton structured source*, viz. a structured source consisting of only one

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[1]This definition of the category **CABAO** contains certain subtle points, which we will explain in a minute.

[2]If we don't restrict to fibre-small constructs, then we need to consider structured sources whose size are *proper classes*. However, this is not a problem for us to worry about. We refer the readers to [1] for more details.

function. This in particular suggests that topological categories are special types of *fibrations* where we can perform lifts against an arbitrary set of morphisms with a common codomain. Together with the uniqueness part of the definition, a topological category satisfies many desirable properties:[3]

Lemma 2.3. *If $\mathscr{A}$ is a topological category, then each fibre $\mathscr{A}_X$ is a complete lattice for any set X.*

Proof. For any family $\{A_i\}_{i \in I}$ in the fibre $\mathscr{A}_X$, consider the structured source, $\{1_X : X \to |A_i|\}_{i \in I}$. It is routine to verify that its unique initial lift is precisely the meet of this family in $\mathscr{A}_X$. □

The existence of initial lifts guarantees each fibre to be complete preorders, and the uniqueness then implies that they are indeed posets. As a fibration, given any function $f : X \to Y$, the initial lifts along f will induce functions of the form $f^* : \mathscr{A}_Y \to \mathscr{A}_X$. Again, f^* being a well-defined function is guaranteed by the uniqueness of initial lifts, and we will also denote maps of the form f^* as *pullback maps*. Furthermore, uniqueness also suggests that the fibration *splits*, in the sense that $1_X^* = 1_{\mathscr{A}_X}$ and $g^* f^* = (gf)^*$. The more important observation is that each pullback map preserves meets in the fibre:

Lemma 2.4. *Let $(\mathscr{A}, |-|)$ be a topological category, then for any function $f : X \to Y$, the pullback map $f^* : \mathscr{A}_Y \to \mathscr{A}_X$ preserves arbitrary meets.*

Proof. For any family $\{B_i\}_{i \in I}$ in $\mathscr{A}_Y$, we only need to prove $\bigwedge_{i \in I} f^* B_i \leq f^* \bigwedge_{i \in I} B_i$. By definition, this holds iff the identity function, viewed as a map $1_X : |\bigwedge_{i \in I} f^* B_i| \to |f^* \bigwedge_{i \in I} B_i|$, is an $\mathscr{A}$-morphism. By the universal property of initial lift, it is so iff $f \circ 1_X = f : |\bigwedge_{i \in I} f^* B_i| \to |\bigwedge_{i \in I} B_i|$ is an $\mathscr{A}$-morphism, and again, this is furthermore equivalent to all the maps in the structured source $\{f : |\bigwedge_{i \in I} f^* B_i| \to |B_i|\}$ being $\mathscr{A}$-morphisms. However, we know that $\bigwedge_{i \in I} f^* B_i \leq f^* B_i$ for any $i \in I$, which means both $1_X : |\bigwedge_{i \in I} f^* B_i| \to |f^* B_i|$ and $f : |f^* B_i| \to |B_i|$ are $\mathscr{A}$-morphisms, hence so is the composite. □

It follows that each pullback map f^* has a unique left adjoint, which we denote as $f_!$ and call it the *pushforward map*. By the adjunction $f_! \dashv f^*$ and the universal property of initial lift, it is easy to see that $f_!$ are exactly describing the cocartesian lifts, which makes a topological category an *opfibration* as well, hence a *bifibration*. As Theorem 2.5 will show, the data of fibres and pullback or pushforward maps uniquely determines a topological category:

Theorem 2.5. *Let **InfL** (resp. **SupL**) be the category of inflattices (suplattices).[4] Recall that they are canonically dual to each other. The data of a topological category $(\mathscr{A}, |-|)$ is the same as the data of a functor $\mathscr{A}_{(-)} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{InfL}$, or equivalently $\mathscr{A}_- : \mathbf{Set} \to \mathbf{SupL}$.*

Proof. We've already shown that a topological category induces a functor from $\mathbf{Set}^{\mathrm{op}}$ to $\mathbf{InfL}$. On the other hand, since $\mathbf{InfL}$ is a subcategory of $\mathbf{Cat}$, any functor $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{InfL}$ admits a Grothendieck construction, resulting in a fibration $p : \mathscr{F} \to \mathbf{Set}$. The objects of $\mathscr{F}$ are pairs (X, A) with A being an element in $F(X)$; a morphism $f : (X, A) \to (Y, B)$ is a function $f : X \to Y$, such that $A \leq Ff(B)$. The forgetful functor p is evident. To this end, we only need to verify that for arbitrary structured source $\{f_i : X \to p(X_i, A_i)\}_{i \in I}$, it has a unique initial lift, which we claim is given by $\bigwedge_{i \in I}(Ff_i)(A_i)$ over X. For any function $g : p(Y, B) \to p(X, \bigwedge_{i \in I}(Ff_i)(A_i))$, by definition it is an $\mathscr{F}$-morphism iff

$$B \leq Fg \bigwedge_{i \in I}(Ff_i)(A_i) = \bigwedge_{i \in I} F(f_i \circ g)(A_i) \Leftrightarrow \forall i \in I[F(f_i \circ g)B \leq A],$$

which exactly means that all $f_i \circ g : p(Y, B) \to p(X_i, A)$ are $\mathscr{F}$-morphisms. Hence, $(\mathscr{F}, p)$ is a topological category, and we leave the readers to verify that the above two processes are mutually inverse. □

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[3]The following two lemmas are both contained in [1]. We include the proof here for the convenience of the readers.

[4]**InfL** (resp. **SupL**) is the category of complete lattices with meet (resp. join) preserving maps. For more detailed description of various categorical structures on **InfL** or **SupL**, we refer the readers to [16, Chapter I].

Proposition 2.6. *All the categories of semantics mentioned in Example 2.1 are topological categories.*

Proof Sketch. It is evident that all the fibres of those mentioned examples are complete lattices. We only describe in each case how the pullback or pushforward maps are constructed, and trust the readers to verify the universal properties and functoriality. Given a function $f : X \to Y$:

- In **Kr**, f^* lifts a relation R on Y to the largest relation in X such that f is monotone, i.e. for any $x, x' \in X$, $(x, x') \in f^*R$ iff $(fx, f'x) \in R$. The pullback maps in **Pre**, **Eqv** are inherited from **Kr**.

- In **Top**, the pullback f^* maps a topology γ on Y to the so-called weak topology on X, i.e. $U \in f^*\gamma$ iff there exists $V \in \gamma$ that $U = f^{-1}(V)$.

- In **Nb**, the description of f^* is similar to that in **Top**. For a neighbourhood relation F on Y, the lift f^*F satisfies that $(x, U) \in f^*F$ iff there exists $V \subseteq Y$ that $U = f^{-1}(V)$ and $(fx, V) \in F$.

- In **CABAO**, it is easier to describe the pushforward maps. Given any endo-function m on $\wp(X)$, its pushforward is the operator $\forall_f \circ m \circ f^{-1}$ on Y, where $\forall_f$ is the right adjoint of f^{-1}.

- In **Evl**, evidently the pullback f^* is obtained by post-composing with f^{-1}. $\square$

At this point, we have accomplished our first goal to recognise all the instances of semantics in Example 2.1 as topological categories. We end this section by describing the *product* construction:

Definition 2.7 (Product of Topological Categories)**.** For any family $\{\mathscr{A}_i\}_{i \in I}$ of topological categories viewed as functors $\{\mathscr{A}_i : \mathbf{Set}^{\mathrm{op}} \to \mathbf{InfL}\}_{i \in I}$, their product $\prod_{i \in I} \mathscr{A}_i$ is given as the following composition,

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{\prod_{i \in I} \mathscr{A}_i} \prod_{i \in I} \mathbf{InfL} \xrightarrow{\oplus_{i \in I}} \mathbf{InfL}.$$

The functor $\oplus_{i \in I}$ is the biproduct functor on **InfL**, which takes a family of inflattices to its set-theoretic product with entry-wise order. In other words, the fibre $(\prod_{i \in I} \mathscr{A}_i)_X$ of a product is simply the product of the fibres $\prod_{i \in I}(\mathscr{A}_i)_X$. It is easy to verify that $\prod_{i \in I} \mathscr{A}_i$ is indeed their categorical product in the category of concrete categories and concrete functors. The product construction for instance allows us to combine a Kripke model with an evaluation function by looking at $\mathbf{Kr} \times \mathbf{Evl}$, or to consider a family of models by introducing $\mathscr{A}^\Sigma$ for any set Σ, which is the Σ-indexed product of $\mathscr{A}$ with itself.

3   Interpreting Modalities via Geometric Data

In this section, we will see how the categorical structure we have described in Section 2 would unify the *interpretation of modalities* in each different types of semantics. We start by briefly recalling the very basics of the modal language and its interpretation; standard references include [11, 7]. Let a non-empty set Σ serve as the signature, and let P be a non-empty set of propositional variables. The modal language $\mathscr{L}_\Sigma$ over the signature Σ and the variable set P is the smallest set of formulas containing P and closed under forming conjunctions, negations, and adding modalities $\square_a$ for all $a \in \Sigma$. When Σ is a singleton, we will omit the subscript, and $\mathscr{L}$ denotes the usual modal language with a single modality. We will refer to it as the *basic modal language*. Other logical connectives are viewed as defined notions.

In any set-based semantics of modal logic, the classical propositional connectives are always interpreted by the Boolean operations on the power set algebra. From an algebraic point of view, the interpretation of the additional modality, in its most general form, should be given by an arbitrary *endo-function on the power set*, which is exactly the structure of a CABAO. Hence, we define the structure of a *semantic functor* to provide the interpretation of basic modal language:

Definition 3.1 (Semantic Functor and Modal Category)**.** Let $(\mathscr{A}, |-|)$ be a topological category. A *semantic functor* on $\mathscr{A}$ is a concrete functor $(-)^+ : \mathscr{A} \to \mathbf{CABAO}$. A *modal category* is then a topological category together with a semantic functor.

For any modal category $\mathscr{A}$ with semantic functor $(-)^+$, we recursively define the interpretation of modal formulas as follows: For any set X and any pair (A, V) in $(\mathscr{A} \times \mathbf{Evl})_X$,

$$\llbracket p \rrbracket_A^V = V(p), \quad \llbracket \varphi \wedge \psi \rrbracket_A^V = \llbracket \varphi \rrbracket_A^V \cap \llbracket \psi \rrbracket_A^V, \quad \llbracket \neg \varphi \rrbracket_A^V = X \backslash \llbracket \varphi \rrbracket_A^V, \quad \llbracket \Box \varphi \rrbracket_A^V = A^+ (\llbracket \varphi \rrbracket_A^V).$$

We may also define the more familiar *local* version of semantics, and write $A, V, x \models \varphi$ whenever $x \in \llbracket \varphi \rrbracket_A^V$. Evidently, the identity functor on $\mathbf{CABAO}$ establishes itself as a modal category. We see below that all other categories of semantics mentioned previously have modal category structures:

Proposition 3.2. *There exist* fully faithful *modal functors on* $\mathbf{Kr}, \mathbf{Pre}, \mathbf{Eqv}, \mathbf{Top}$ *and* $\mathbf{Nb}$ *that embeds them into* $\mathbf{CABAO}$*, inducing the usual semantics of modal logic.*

Proof Sketch. Again, we only describe the construction of semantic functors in each case, and trust the readers to verify their fully faithfulness:

- Recall for any relation $R \subseteq X \times Y$, there exists an induced operator $\forall_R : \wp(X) \to \wp(Y)$, such that for any $S \subseteq X$, $\forall_R(S) = \{ y \in Y \mid \forall x[xRy \Rightarrow x \in S] \}$. We then construct the embedding $\mathbf{Kr} \hookrightarrow \mathbf{CABAO}$ by sending each relation R in fibre $\mathbf{Kr}_X$ to the operator $\forall_{R^\dagger}$, where $R^\dagger$ is the dual relation of R. The semantic functors on $\mathbf{Pre}$ and $\mathbf{Eqv}$ are inherited from the one on $\mathbf{Kr}$.

- For $\mathbf{Top}$, it sends each topology τ on a set X to the interior operator j_τ it induces.

- For $\mathbf{Nb}$, it assigns E in $\mathbf{Nb}_X$ to n_E, such that $n_E(S) = \{ x \mid (x, S) \in E \}$ for any $S \subseteq X$.   □

Proposition 3.2 then completes our categorical unification of all the mentioned types of semantics on how they interpret the basic modal language. Clearly, our approach of given in Definition 3.1 closely relates to the spirit of *algebraic semantics* of modal logic. But one additional insight our categorical framework suggests is an even closer connection between these different types of semantics with modal algebras via Proposition 3.2, in that the single notion of continuous morphisms between CABAOs as defined in Example 2.1 explains all the different types of morphisms in these topological categories, by identifying them as *full subcategories* of $\mathbf{CABAO}$.

Intuitively, it is precisely the semantic functor that provides the interpretation of modalities in all cases, but we can establish the correspondence in a more formal way, by considering *transformation of models* as mentioned in Section 1. We define when a concrete functor between two modal categories interacts well with a specific fragment of modal logic:

Definition 3.3 (Preservation of Language)**.** Let $\mathscr{A}, \mathscr{B}$ be two modal categories, which both support the interpretation of certain fragment of modal language $\mathscr{L}_0$ which extends $\mathscr{L}$. We say a concrete functor $F : \mathscr{A} \to \mathscr{B}$ *preserves the interpretation of the language* $\mathscr{L}_0$, if the following happens: For (A, V) in $(\mathscr{A} \times \mathbf{Evl})_X$ over some set X and for any formula $\varphi \in \mathscr{L}_0$, we have $\llbracket \varphi \rrbracket_A^V = \llbracket \varphi \rrbracket_{FA}^V$.

In other words, a concrete functor F preserves the interpretation of a language $\mathscr{L}_0$ iff the evaluation of each formula in $\mathscr{L}_0$ remains unchanged when we apply the transformation F. As a first example of establishing an exact correspondence between a semantic structure and a particular syntactic pattern, we prove the following theorem:

Theorem 3.4. *For any concrete functor $F : \mathscr{A} \to \mathscr{B}$ between two modal categories $(\mathscr{A}, (-)_{\mathscr{A}}^+)$ and $(\mathscr{B}, (-)_{\mathscr{B}}^+)$, it commutes with the two semantic functors iff it preserves the interpretation of $\mathscr{L}$.*

Proof. Suppose F does not commute with the two semantic functors, then for some object A in $\mathscr{A}$ over some set X, $(A)^+_{\mathscr{A}}$ and $(FA)^+_{\mathscr{B}}$ would not agree. This means that the two operators on $\wp(X)$ do not coincide, which implies they must not coincide on some subset $S \subseteq X$. Consider the simple formula $\Box p$, and an evaluation function V that assigns p to S. By definition, $[\![\Box p]\!]^V_A$ and $[\![\Box p]\!]^V_{FA}$ will *not* be the same.

The proof of the only if direction is obviously by induction on the structure of formulas, and the only interesting case is the one involving modalities. Since F is assumed to be a modal functor, we must have $(A)^+_{\mathscr{A}} = (FA)^+_{\mathscr{B}}$ for any A in $\mathscr{A}$, which means that the interpretation of the modalities by A through $(-)^+_{\mathscr{A}}$ and by FA through $(-)^+_{\mathscr{B}}$ are identical, which suffices for the inductive proof. □

Theorem 3.4 provides the precise formal content of what we mean informally by the correspondence between the syntactic structure of modalities and the semantic structure of semantic functors of a modal category. And henceforth, we will refer to those concrete functors between two modal categories which commutes with the semantic functors on both sides as *modal functors*. There are already many interesting examples of modal functors we can explore, and below we only list a few:

Example 3.5. Here we list some interesting examples of model transformations between the modal categories we have introduced so far:

- By definition, any modal category has a unique modal functor mapping into **CABAO**, which makes it the terminal object in the category of modal categories and modal functors.

- Since the semantic functors in **Pre** and **Eqv** are induced by the one in **Kr**, the embeddings **Eqv** ↪ **Pre** and **Pre** ↪ **Kr** are both modal functors.

- There is a modal embedding **Pre** ↪ **Top**, assigning a preorder its Alexandroff topology.

- In fact, we can show that **Nb** is isomorphic to **CABAO**, which means that all the above examples has a modal embedding into **Nb** as well.

It is also instructive to look at counter-examples of modal functors. It turns out, the above modal embeddings all have either a left or a right adjoint, and these adjoints are usually *not* modal embeddings with respect to the semantic functors we have constructed in Proposition 3.2:

- We have both a left and a right adjoint **Pre** $\rightrightarrows$ **Eqv** for the modal embedding **Eqv** ↪ **Pre**, sending a preorder to the smallest equivalence relation containing it and the least one it contains. These adjoints do not commute with the semantic functors since they change the relation. Similarly, there is a left adjoint **Kr** → **Pre** sending a relation to its preorder closure, which isn't modal either.

- The embedding **Pre** ↪ **Top** has a right adjoint **Top** → **Pre**, sending a topological space to its specialisation order, but this construction does not preserve the information of all open neighbourhoods of a point, hence it is also not modal. ◇

However, the mere syntactic structure of a modality, arguably, has not too much to do with the rich structure of topological categories we have seen in Section 2. In fact, the notion of semantic functors and modal categories in Definition 3.1 can indeed be stated more generally for concrete categories, not only for topological ones. The true usage of the full structure of topological categories emerges when we consider further syntactic extensions of modal logic, which are the topics of the next two sections.

4   Modal Strength, Group Knowledge and Fibre Structure

In this section, we will proceed to study the extension of multi-agent fragment of modal logic, with explicit syntactic comparison of *modal strength*, or *dependence relation*, between different modalities,

and forming *group agents*. Recent works [3, 2] put dependence purely in modal terms, but they have only considered the relational and topological contexts. How to form group agents is also an active topic for current research on modal logic and collective agency [13, 23], but almost all approaches focus on a single type of models. In both cases, our categorical approach allows a unifying description for all types of semantics, which is one of the main benefit. Our ultimate goal is again to identify an exact correspondence between these syntactic patterns with certain semantic structures of topological categories, with formal content similar to that of Theorem 3.4.

Let's first look at the simple extension of a multi-modal language, i.e. when the indexed set Σ is not a singleton. There will be different modalities $\Box_a, \Box_b, \cdots$ with $a, b \in \Sigma$ in the language $\mathscr{L}_\Sigma$. It should be straight forward to recognise that the multi-agent fragment $\mathscr{L}_\Sigma$ are related to taking the *products* of topological categories. Given any modal category $\mathscr{A}$, recall that we use $\mathscr{A}^\Sigma$ to denote the Σ-indexed self-product of $\mathscr{A}$. Any semantic functor $(-)^+$ on $\mathscr{A}$ naturally extends to one from $\mathscr{A}^\Sigma$ to **CABAO**$^\Sigma$, which by an abuse of notation we also denote as $(-)^+$: Given any object $(A_a)_{a\in\Sigma}$ in the fibre $\mathscr{A}^\Sigma_X$, which by our construction in Definition 2.7 is simply a Σ-indexed tuple of objects in the fibre $\mathscr{A}_X$, we have $(A_a)^+_{a\in\Sigma} = (A^+_a)_{a\in\Sigma}$. The Σ-indexed tuple $(A^+_a)_{a\in\Sigma}$ is then expected to provide the interpretation of each modality $\Box_a$ in the language $\mathscr{L}_\Sigma$ for any $a \in \Sigma$, using the corresponding object A^+_a. Intuitively, different modalities correspond to different objects in the same fibre of a topological category. Hence, given any $((A_a)_{a\in\Sigma}, V)$ in the fibre $(\mathscr{A}^\Sigma \times \mathbf{Evl})_X$, we may change the clause of modalities in the recursive definition of evaluation of formulas to $[\![\Box_a\varphi]\!]^V_{(A_a)_{a\in\Sigma}} = (A_a)^+([\![\varphi]\!]^V_{(A_a)_{a\in\Sigma}})$, to interpret $\mathscr{L}_\Sigma$.

However, in the language $\mathscr{L}_\Sigma$, we treat different modalities as different individuals, and do not consider the possible relations between different modalities. But we do have a meaningful way comparing them, since semantically they denote different objects within the same fibre of a topological category $\mathscr{A}$, and there is a canonical order in each fibre $\mathscr{A}_X$. It turns out, this partial order within each fibre signifies the *modal strength* of different modalities. Explicitly, suppose we have two objects A, B in the fibre $\mathscr{A}_X$ that $A \leq B$. The semantic functor then gives us two operators $m_A \leq m_B$ in **CABAO**$_X$, which, according to our definition of morphisms in **CABAO**, actually means $m_B \subseteq m_A$.

In different contexts, the modal strength relation has various incarnations. For instance, in epistemic or doxastic logic, we read the modal formula $\Box_a\varphi$ as agent-a knows or believes φ (cf. [10]). Now if we have $A_a \leq A_b$ in the fibre $\mathscr{A}_X$, the above induced two modalities satisfying $m_b \subseteq m_a$ would actually suggest that there is an *epistemic dependence* between the two agents' knowledge or belief: Whenever b knows some proposition at state $x \in X$, viz. $x \in m_b([\![\varphi]\!])$, a also knows it at that state, because $x \in m_b([\![\varphi]\!]) \subseteq m_a([\![\varphi]\!])$. In other applications, such modal strength comparison would mean something else.

This observation motivates us to add such comparison of modalities explicitly into our syntax, in the form of *dependence atoms*. For any $a, b \in \Sigma$, we could add an atomic proposition $K_a b$ into our language, with the intuitive reading of $K_a b$ as stating the modality denoted by a lies below the one denoted by b. We refer to this extended language as $\mathscr{L}^D_\Sigma$. But to interpret such dependence atoms as predicates, we need the following *local* version of strength orders between two operators on the same power set algebra:

Definition 4.1. For any two operators m, n in **CABAO**$_X$ and any $U \subseteq X$, we say m *locally depends on n in U*, denoted as $m \subseteq_U n$, if for any $S \subseteq X$ and any $x \in U$, $x \in m(S) \Rightarrow x \in n(S)$.

In this way, the global relation $m \subseteq n$ is the same as $m \subseteq_X n$. When U is a singleton $\{x\}$, we simply write $m \subseteq_x n$. The following observation is crucial for us to define the interpretation of the dependence atoms:

Lemma 4.2. *For any m, n in **CABAO**$_X$, there is a maximal subset U that $m \subseteq_U n$.*

Proof. By definition, for the empty set $\emptyset$ we always have $m \subseteq_\emptyset n$, since the universal quantification $\forall x \in \emptyset$ is vacuous. Furthermore, local dependence is closed under taking unions, since it is trivial to note that $m \subseteq_{\bigcup_{i\in I} U_i} n$ iff for any $i \in I$, $m \subseteq_{U_i} n$. Thus, the maximal subset U is given by $\{x \mid m \subseteq_x n\}$. $\square$

Given an object $(A_a)_{a \in \Sigma}$ in the fibre $\mathscr{A}_X^\Sigma$, the interpretation $[\![K_a b]\!]$ of the newly added dependence atoms should now be defined as the maximal subset U of X, such that $A_b^+ \subseteq_U A_a^+$ holds. This is exactly how the dependence atoms are interpreted in any topological categories. We might also give the local version of the truth condition, $(A_a)_{a \in \Sigma}, x \models K_a b$ iff $A_b^+ \subseteq_x A_a^+$. Notice that the interpretation of $K_a b$ is *independent* from the choice of the evaluation function V on X. We may look at the concrete meaning of such dependences in all the remaining examples we have considered so far:

Example 4.3. We list here how local dependence looks like in each exemplar modal category:

- In **Kr**, **Pre** and **Eqv**, given relations $(R_a)_{a \in \Sigma}$ on X, we have $(R_a)_{a \in \Sigma}, x \models K_a b$ iff $R_a[x] \subseteq R_b[x]$. In the epistemological interpretation, this means agent-a's uncertainty locally at x is less than b's.

- In **Top**, given topologies $(\tau_a)_{a \in \Sigma}$ on X, $(\tau_a)_{a \in \Sigma}, x \models K_a b$ iff $1_X : (X, \tau_a) \to (X, \tau_b)$ is *locally continuous* at x. This relates to the continuity view of epistemic dependence discussed in [2].

- In **Nb**, given neighbourhoods $(E_a)_{a \in \Sigma}$ on X, $(E_a)_{a \in \Sigma}, x \models K_a b$ iff $E_b[x] \subseteq E_a[x]$. In evidence based logic, this interprets as the evidence set of b's is contained in that of a's locally at x (cf. [9]). ◇

In this case, preserving the interpretation of the multi-agent modalities and the dependence atoms does not require anything else than being a modal functor:

Theorem 4.4. *For any concrete functor $F : \mathscr{A} \to \mathscr{B}$ between two modal categories, it preserves the interpretation of $\mathscr{L}_\Sigma^D$ iff it preserves the interpretation of $\mathscr{L}$.*

Proof. The only if part is trivial, since $\mathscr{L}_\Sigma^D$ is an extension of $\mathscr{L}$. For the if part, by Theorem 3.4 we know F must be a modal functor. This implies that for any $a \in \Sigma$ and any tuple $(A_a)_{a \in \Sigma}$ in $\mathscr{A}^\Sigma$, we must have $(A_a)_{\mathscr{A}}^+ = (FA_a)_{\mathscr{B}}^+$, which means that A_a induces the same operator as FA_a. This suffices for the preservation of the fragment $\mathscr{L}_\Sigma$ by F. F preserving dependence atoms is also immediate, since their interpretation only relies on the operators on the underlying set. □

However, this changes once we start to combine sets of agents into a single agent and consider such group structures explicitly in our syntax. From a philosophical perspective, when modelling the inference and reasoning patterns of agents under certain information structure using modal logic, we not only care about individual agents themselves, but we would also like to study how *a group of agents* as a whole reasons and interacts with each other. As mentioned, this is an active topic on how to represent group agency in different contexts. Most of the traditional developments of group agency in modal logic are based on Kripkean semantics [7, 8], but there has been recent efforts exploring how to define common knowledge of a group in topological semantics [4]. Again, our categorical approach would uniformly describe the group structure in any topological category associated with every type of semantics.

To combine a group of agents to a single one, it requires us to transform an object in $\mathscr{A}^G$ for any subset $G \subseteq \Sigma$, which is a tuple representing each individual agent in the group G, to a single object in $\mathscr{A}$, which corresponds to the collective group agent. Naturally, there are two canonical ways to do this in general for any set G, using the fact that each fibre in a topological category is not only a poset, but indeed a *complete lattice*. In particular, we can form two (families of) concrete functors $\bigwedge, \bigvee : \mathscr{A}^G \to \mathscr{A}$. As the symbols suggest, for any tuple $(A_a)_{a \in G}$ in $\mathscr{A}^G$, they act on it as follows: $\bigwedge (A_a)_{a \in G} = \bigwedge_{a \in G} A_a$, and $\bigvee (A_a)_{a \in G} = \bigvee_{a \in G} A_a$. Functoriality of $\bigwedge, \bigvee$ should be immediate.

These functors then allow us to combine a group of agents of arbitrary size into a single one. We will denote them as the $\bigwedge$- and $\bigvee$-combination of group agents, and they correspond to two different readings of what a group of agents means. Intuitively, the $\bigwedge$-combination means the group *shares the information* of each individual, as if they are *physically together*. Because once we form a group $\bigwedge_{a \in G} A_a$, for any

individual a in the group G we would have $\bigwedge_{a \in G} A_a \leq A_a$ in the fibre, which implies $A_a^+ \subseteq (\bigwedge_{a \in G} A_a)^+$. Just as we have discussed before, if we adopt an epistemic interpretation of modalities, this means that whatever agent a knows, so does the group, and this holds for any agent in this group. Furthermore, the meet taken in the fibre $\mathscr{A}_X$ actually shows that the group modelled by $\bigwedge_{a \in \Sigma} A_a$ is the universal one that has this property. This informally suggests that the group acts like an agent who has access to all the information owned by each individual agent in this group, exactly like the case when everyone in the group has come to a single location, and put all of their information on the table where anyone can see. In **Pre** or **Eqv**, the $\bigwedge$-combination simply take the conjunction of all the relations, and this is exact the well-known *distributive knowledge* of a group (cf. [8]). Hence, the $\bigwedge$-combination generalises distributive knowledge to all types of semantics.

On the other hand, the $\bigvee$-combination means the group *shares the uncertainties* of each individual, as if they are only *abstractly* considered as a single agent. Dual to the case before, we must have $(\bigvee_{a \in G} A_a)^+ \subseteq A_a$ for any $a \in G$. This implies that for the combined group, if it knows something then necessarily each individual in the group also knows this, and the group agent is the universal one that has this property. To better compare with the existing literature, we observe the following simple result:

Lemma 4.5. *If the semantic functor $(-)^+$ on a topological category $\mathscr{A}$ always induces monotone and idempotent operators, then $(\bigvee_{a \in G} A_a)^+ \subseteq A_{a_1}^+ \circ \cdots \circ A_{a_n}^+$ for any $a_1, \cdots, a_n \in G$.*

Proof. If follows by $(\bigvee_{a \in G} A_a)^+ \subseteq A_{a_i}^+$ for any i, and monotonicity, idempotence of these operators. $\square$

Translating back to natural language, in the condition of Lemma 4.5, what the $\bigvee$-combined group knows is much more restrictive, in that if the group knows something, then any agent in the group also knows it, and furthermore a_i knows that a_j knows that $\cdots$ that a_k knows it. This shows that the $\bigvee$-combination is a generalisation of the *common knowledge* of a group (again, cf. [8]).

We may now formally define the syntactic extension where we also allow group formation in our logic. For any indexed set Σ, we let Σ_l, Σ_r be *synonyms* for the power set $\wp(\Sigma)$. The language $\mathscr{L}_{\Sigma_l}^D$ and $\mathscr{L}_{\Sigma_r}^D$ is nothing more but the modal languages with agent symbols in Σ_l, Σ_r, respectively, together with all the dependence atoms between these group agents. However, we write in this way because to interpret the language $\mathscr{L}_{\Sigma_l}^D$ or $\mathscr{L}_{\Sigma_r}^D$, we still only need to work within $\mathscr{A}^\Sigma$, not $\mathscr{A}^{\Sigma_l}$ or $\mathscr{A}^{\Sigma_r}$.

Given an object $(A_a)_{a \in \Sigma}$ in $\mathscr{A}^\Sigma$ over the set X, we can interpret the modal operators for a group of agents in the two fragments as either the $\bigwedge$- or $\bigvee$-combination. For any subset $G \subseteq \Sigma$, we define the interpretation of $\square_G$ in $\mathscr{L}_{\Sigma_l}^D$ as the operator $(\bigwedge_{a \in G} A_a)^+$; and similarly for the language $\mathscr{L}_{\Sigma_r}^D$, $\square_G$ is interpreted as the operator $(\bigvee_{a \in G} A_a)^+$. Building on what we have developed before, this suffices to interpret the two languages $\mathscr{L}_{\Sigma_l}^D$ and $\mathscr{L}_{\Sigma_r}^D$. Of course, for a singleton group $\{a\}$, its interpretation under the two fragments coincide, which still corresponds to the usual interpretation of the operator A_a^+. The upshot is that we can identify the following valid logical rules in the two fragments $\mathscr{L}_{\Sigma_l}^D$ and $\mathscr{L}_{\Sigma_r}^D$:

Proposition 4.6. *For any modal category $\mathscr{A}$, the following axioms are valid in $\mathscr{L}_{\Sigma_l}^D$ (resp. $\mathscr{L}_{\Sigma_r}^D$):*[5]

- **Inclusion***: $K_G H$ (resp. $K_H G$), provided $H \subseteq G$;*

- **Additivity***: $K_G H \wedge K_G P \rightarrow K_G(H \cup P)$ (resp. $K_H G \wedge K_P G \rightarrow K_{H \cup P} G$);*

- **Transitivity***: $K_G H \wedge K_H P \rightarrow K_G P$ (resp. $K_G H \wedge K_H P \rightarrow K_G P$);*

- **Transfer***: $K_G H \wedge \square_H \varphi \rightarrow \square_G \varphi$ (resp. $K_G H \wedge \square_H \varphi \rightarrow \square_G \varphi$).*

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[5]Half of these axioms corresponding to the fragment $\mathscr{L}_{\Sigma_l}^D$ has already been identified in [3, 2] in the special case of **Top**.

Proof. We only prove the case for $\mathscr{L}^D_{\Sigma_l}$; the other case is completely dual. Let $(A_a)_{a\in\Sigma}$ be any object in $\mathscr{A}^\Sigma$ over X. Whenever we have $H \subseteq G \subseteq \Sigma$, we have $A_G = \bigwedge_{a\in G} A_a \subseteq \bigwedge_{a\in H} A_a = A_H$, which implies $A_H^+ \subseteq A_G^+$. Hence, according to our definition of the interpretation of the dependence atoms, we have $[\![K_G H]\!] = X$, and this validates **Inclusion**. For any two groups H, P, by definition $A_{H\cup P} = \bigwedge_{a\in H\cup P} A_a = A_H \wedge A_P$, which implies $m_H \cup m_P \subseteq m_{H\cup P}$. Now locally, suppose for some $x \in X$ we have $x \in [\![K_G H]\!]$ and $x \in [\![K_G P]\!]$. Then for any $S \subseteq X$, $x \in m_{H\cup P}(S) \Rightarrow x \in m_H(S) \cup m_P(S)$. Either $x \in m_H(S)$ or $x \in m_P(S)$, we would have $x \in m_G(S)$, according to our assumption that $K_G H$ and $K_G P$ locally holds at x. Hence, the **Additivity** law also holds. The validity of **Transitivity** and **Transfer** axioms are evident. $\qquad\square$

Up to this point, we have completed our generalisation of group structure to all the exemplar modal categories in a uniform way, and identified a set of valid inference rules. The remaining task is then to identify which part of the semantic structure in topological categories does the syntactic group-forming operation corresponds to. Considering our usage of the complete lattice structure of fibres, the following result should be of no surprise:

Theorem 4.7. *Let $F : \mathscr{A} \to \mathscr{B}$ be a modal functor between two modal categories, and suppose the semantic functor $(-)^+_{\mathscr{B}}$ is injective on objects. F preserves arbitrary meets (resp. joins) fibre-wise, i.e. the induced functions $F_X : \mathscr{A}_X \to \mathscr{B}_X$ on fibres is a morphism in* **InfL** *(resp.* **SupL***) for any set X, iff it preserves the interpretation of the language $\mathscr{L}^D_{\Sigma_l}$ (resp. $\mathscr{L}^D_{\Sigma_r}$) for any indexed set Σ.*

Proof. Again, we only prove the case for F preserving meets fibre-wise and the preservation of the interpretation of $\mathscr{L}^D_{\Sigma_l}$. We already know from Theorem 4.4 that F is a modal functor iff it preserves the interpretation of $\mathscr{L}^D_\Sigma$, thus it suffices to show it further preserves the interpretation of $\bigwedge$-group-formation iff it preserves meets fibre-wise. From how the $\bigwedge$-group modality is defined, it is immediate to note that F preserves the interpretation of $\mathscr{L}^D_{\Sigma_l}$ iff $(\bigwedge_{a\in\Sigma} A_a)^+_{\mathscr{A}}$, which by the fact of F being a modal functor is equal to $(F \bigwedge_{a\in\Sigma} A_a)^+_{\mathscr{B}}$, coincides with $(\bigwedge_{a\in\Sigma} FA_a)^+_{\mathscr{B}}$, for any $(A_a)_{a\in\Sigma}$. By assumption on $(-)^+_{\mathscr{B}}$, this holds iff $F \bigwedge_{a\in\Sigma} A_a = \bigwedge_{a\in\Sigma} FA_a$, which exactly means F preserves meets fibre-wise. $\qquad\square$

Consider the various model transformations we have described in Example 3.5, Theorem 4.4 immediately tells us how these functors behave with respect to group knowledge. For instance, since the modal embedding **Eqv** $\hookrightarrow$ **Pre** has both a concrete left and right adjoint, it must preserve both meets and joins fibre-wise, which suggests that the two fragments $\mathscr{L}^D_{\Sigma_l}$ and $\mathscr{L}^D_{\Sigma_r}$ behave coherently between **Eqv** and **Pre**. However, as for the embedding of **Eqv** and **Pre** into **Kr**, it only has a concrete left adjoint but lacks a right one, which means only the $\bigwedge$-group formation, viz. the distributive knowledge, coincide in **Eqv**, **Pre** and **Kr**, but not the common knowledge. We can see this more explicitly, since the join in fibres of **Kr** are simply unions of relations, while in **Pre** and **Eqv** we must further take the transitive closure of unions of relations. Other modal embeddings can be analysed in a similar fashion.

5 Logical Dynamics and Fibre Connections

The "dynamic turn" of modal logic in the recent two decades makes *logical dynamics* another very important topic in the current literature. In this section, we will see how certain general types of logical dynamics could be subsumed into our categorical framework in a similar fashion as before.

Logical dynamics concerns with the reasoning patterns of agents when *new information* comes in, which generally *changes the underlying set* of a model. This is where the *fibre connection* plays a crucial role, because it allows us to transfer the geometric data over the original model to the updated model

in a uniform way. For simplicity, below we describe all the dynamic extensions based on the simplest fragment $\mathscr{L}$, but it should be clear that our method can be equally applied to other fragments as well.

To warm up, we start by generalising the simplest form of dynamic logic, known as PAL, public announcement logic (cf. [20, 21]). It concerns with information events of publicly announcing that φ holds, which we denote as $!\varphi$. A typical formula in PAL is of the form $[!\varphi]\psi$, intuitively read as ψ is true after announcing φ. For a modal category $\mathscr{A}$, given any object (A,V) in the fibre $(\mathscr{A} \times \mathbf{Evl})_X$, the information event $!\varphi$ naturally restricts the domain X to the subset $S = [\![\varphi]\!]_A^V$. If we denote the inclusion function $S \hookrightarrow X$ as i, then the natural way to transfer the geometric data on X to S is by pulling back along i. This way, we obtain a new semantic model (i^*A, i^*V) over S, and the formula following the dynamic operator $[!\varphi]$ could be interpreted in this new model. We also need to transfer subsets of S back to subsets of X, to maintain the recursive structure of adding dynamic operators within the syntax. The natural candidates are $\exists_i$ and $\forall_i$, which we will see correspond to the pair of dual operators $\langle!\varphi\rangle$ and $[!\varphi]$.

More formally, we define the extension $\mathscr{L}^{\mathbf{PAL}}$ of $\mathscr{L}$ by the smallest set of formulas containing $\mathscr{L}$ and is closed under forming dynamic formulas of the form $[!\varphi]\psi$, with φ, ψ in $\mathscr{L}^{\mathbf{PAL}}$. Following the above informal idea, we define the interpretation of formulas in $\mathscr{L}^{\mathbf{PAL}}$ by adding the following recursive clause: For (A,V) in $(\mathscr{A} \times \mathbf{Evl})_X$, we define $[\![[!\varphi]\psi]\!]_A^V = \forall_i [\![\psi]\!]_{i^*A}^{i^*V}$, and $[\![\langle!\varphi\rangle\psi]\!]_A^V = \exists_i [\![\psi]\!]_{i^*A}^{i^*V}$, where i is the inclusiong map $[\![\varphi]\!]_A^V \hookrightarrow X$. Perhaps the more familiar form of truth conditions of these dynamic operators are the following equivalent local formulation: For any $x \in X$,

$$A,V,x \models [!\varphi]\psi \Leftrightarrow A,V,x \models \varphi \text{ implies } i^*A, i^*V, x \models \psi,$$
$$A,V,x \models \langle!\varphi\rangle\psi \Leftrightarrow A,V,x \models \varphi \text{ and } i^*A, i^*V, x \models \psi.$$

Again, if we combine this general form of semantics of PAL in any modal category with the special description of pullback maps in $\mathbf{Kr}$ given in the proof of Proposition 2.6, we recover exactly the usual PAL dynamics developed for Kripke models, but we also get the PAL dynamics in other types of semantics at the same time. This again exhibits the usefulness of a unifying description of semantics of modal logic.

Expectedly, the syntactic PAL dynamic operators in the $\mathscr{L}^{\mathbf{PAL}}$ fragment should correspond to the semantic structure of initial lifts along inclusions in a topological category:

Theorem 5.1. *Let $F : \mathscr{A} \to \mathscr{B}$ be a modal functor between two modal categories, and suppose the semantic functor $(-)_{\mathscr{B}}^+$ is injective on objects. F further preserves the interpretation of $\mathscr{L}^{\mathbf{PAL}}$ iff it preserves the initial lifts of any injections, i.e. for any inclusion map $i : S \hookrightarrow X$ and for any object A in the fibre $\mathscr{A}_X$, $Fi^*A = i^*FA$ holds.*

Proof. Again for the if direction we prove by induction, and the only case we need to think about is for the PAL dynamic operator. Given φ, ψ and any (A,V) in $(\mathscr{A} \times \mathbf{Evl})_X$, by induction hypothesis we have $[\![\varphi]\!]_A^V = [\![\varphi]\!]_{FA}^V$, and we denote the inclusion map of this subset into X by i. Now by definition of the interpretation of $[!\varphi]\psi$, we have $[\![[!\varphi]\psi]\!]_A^V = \forall_i [\![\psi]\!]_{i^*A}^{i^*V} = \forall_i [\![\psi]\!]_{Fi^*A}^{i^*V} = \forall_i [\![\psi]\!]_{i^*FA}^{i^*V} = [\![[!\varphi]\psi]\!]_{FA}^V$. Thus, F preserves the interpretation of $\mathscr{L}^{\mathbf{PAL}}$.

On the other hand, suppose for some object A in the fibre $\mathscr{A}_X$ and for some injection $i : S \hookrightarrow X$, we have i^*FA is *not* equal to Fi^*A. This in particular suggests that the associated operators $(i^*FA)_{\mathscr{B}}^+$ and $(Fi^*A)_{\mathscr{B}}^+$ on S are not identical, and they must disagree at some subset T of S. Then let V be an interpretation on X such that $V(p) = S$ and $V(q) = T$. Consider the interpretation of the formula $\langle!p\rangle\Box q$. On one hand, we have $[\![\langle!p\rangle\Box q]\!]_A^V = \exists_i [\![\Box q]\!]_{i^*A}^{i^*V} = \exists_i [\![\Box q]\!]_{Fi^*A}^{i^*V} = (Fi^*A)_{\mathscr{B}}^+(T)$. On the other hand, we have $[\![\langle!p\rangle\Box q]\!]_{FA}^V = \exists_i [\![\Box q]\!]_{i^*FA}^{i^*V} = (i^*FA)_{\mathscr{B}}^+(T)$. By assumption, $(Fi^*A)_{\mathscr{B}}^+(T)$ does not coincide with $(i^*FA)_{\mathscr{B}}^+(T)$, and thus F does not preserve the interpretation of $\mathscr{L}^{\mathbf{PAL}}$ by definition. $\qquad\square$

For those model transformations that has a concrete left adjoint, they automatically commutes with all pullback maps, hence preserves the interpretation of $\mathscr{L}^{\mathbf{PAL}}$. Perhaps surprisingly, all of the modal embeddings described in Example 3.5 actually do commutes with pullbacks of *injections*, though not all of them have a concrete left adjoint, and this statement for arbitrary functions is false. As a result, $\mathscr{L}^{\mathbf{PAL}}$ is a particularly nice fragment of dynamic logic to work with.

However, PAL as dynamic logic is still too restrictive. A much more powerful dynamic mechanism is *product update* in DEL, dynamic epistemic logic [6, 18]. In product update, information events themselves form a model E, which carries additional geometric data signifying agent's uncertainly about which event actually happens, and the update is parametrised by E. Each event $e \in E$ is also equipped with a formula φ_e that specifies the precondition of that event happens. For any model over a set X, the updated model is a subset of the product space $E \times X$, consisting of those pairs (e, x) where x satisfies the precondition of e. The geometric data over the updated set takes into account the ones on both X and E.

There are already several categorical reformulation and generalisation of DEL in the literature, e.g. see [17, 12], but most of them are based on relational semantics, while our approach applies to arbitrary topological categories. We first define the notion of a *product type*, which generalises event models:

Definition 5.2 (Product Type). A *product type* E for the modal category $\mathscr{A}$ is a tuple $\langle E, B, W, \{\psi_e\}_{e \in E} \rangle$, where E is a set, and B, W are objects in the fibre $\mathscr{A}_E, \mathbf{Evl}_E$. The family $\{\psi_e\}_{e \in E}$ is an E-indexed family of formulas within the language $\mathscr{L}$.

The notion of *product type update* we are going to describe, which generalises DEL, is parametrised by such a product type E. For any semantic model (A, V) in the fibre $(\mathscr{A} \times \mathbf{Evl})_X$, we write $E \otimes_V X$ as the underlying set of the updated model, which is given by the dependent sum $\sum_{e \in E} [\![\psi_e]\!]_A^V$. Intuitively, the updated model is indexed by events in E, whose fibre over e is the set of all possible words satisfying the precondition ψ_e. There are then two natural projection maps $\pi_X : E \otimes_V X \to X$ and $\pi_E : E \otimes_V X \to E$, and we define the geometric data $(\mathsf{E} \otimes_V A, W \otimes V)$ in the fibre $(\mathscr{A} \times \mathbf{Evl})_{E \otimes_V X}$ to be $\pi_X^* A \wedge \pi_E^* B$ and $\pi_X^* V \wedge \pi_E^* W$, respectively. A typical dynamic formula in product type update is of the form $[\mathsf{E}, S]\varphi$ or $\langle \mathsf{E}, S \rangle \varphi$, where E is a product type and S is a subset of E. We define their interpretation as follows,

$$[\![[\mathsf{E}, S]\Phi]\!]_A^V := \forall_{\pi_X} \left((S \otimes_V X) \to [\![\Phi]\!]_{E \otimes_V A}^{W \otimes V} \right), \quad [\![\langle \mathsf{E}, S \rangle \Phi]\!]_A^V := \exists_{\pi_X} \left((S \otimes_V X) \cap [\![\Phi]\!]_{E \otimes_V A}^{W \otimes V} \right),$$

where the set $S \otimes_V X = \sum_{e \in S} [\![\psi_e]\!]_A^V$ is a subset of $E \otimes_V X$, and $\to, \cap$ are calculated in the power set $\wp(E \otimes_V X)$. Again, interpreted our general construction back in the relational context $\mathbf{Kr}$ of Kripke models, one immediately recovers the usual product update in DEL.[6]

In a word, the way we associate the geometric data on the updated model $E \otimes_V X$ is by pulling back the ones over X and E along the two projection maps, and then take their intersection in the fibre. However, a categorically minded reader would perhaps wonder what happens to the degenerate case where we have an *empty intersection*. Though being kind of trivial, this is in fact important for correspondence results of product type update, which will be stated later. Hence, we also introduce *empty product update*, whose syntactic structure is extremely simple: It is of the form $\mathsf{U}\varphi$, and for any (A, V) in $(\mathscr{A} \times \mathbf{Evl})_X$ we define $[\![\mathsf{U}\varphi]\!]_A^V$ to be $[\![\varphi]\!]_{\top_X}^V$, where $\top_X$ is the maximal element in $\mathscr{A}_X$. This is indeed a form of dynamics, since the operators U results in the change of the geometric data, though the update is constant in all cases. We then define $\mathscr{L}^{\mathbf{PRO}}$ to be the least fragment containing $\mathscr{L}$, which is also closed under taking dynamic formulas of empty product update and product type update.

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[6]In the literature, only the case where S is a singleton set $\{e\}$ is usually considered, but this is a minor generalisation. It is possible to define product update more generally along any function mapping into E, but we leave that for future work.

Now that product type update is properly generalised to arbitrary topological categories, we can realise PAL dynamics as special case of product type update. In fact, for any formula φ, we can associate it with a product type, which we also denote as $!\varphi$. Explicitly, $!\varphi$ is the tuple $\langle 1, \top, \top, \{\varphi\}\rangle$, where 1 is the singleton set, and φ is the corresponding precondition of the single element in 1. It is evident that the updated model by this product type $!\varphi$ is exactly the one obtained by publicly announcing φ in PAL dynamics. In fact, many other types of dynamics turn out to be special cases (cf. [6]).

Now, it should certainly be expected that the dynamic extension $\mathscr{L}^{\mathbf{PRO}}$ corresponds exactly to pull-back maps between fibres and finite meets within fibres. However, for product type update, we need to slightly modify our definition of preservation of languages, since now in the syntax of $\mathscr{L}^{\mathbf{PRO}}$, we have explicitly included certain semantic data, viz. the product types E. We now say a concrete functor $F : \mathscr{A} \to \mathscr{B}$ preserves the interpretation of $\mathscr{L}^{\mathbf{PRO}}$ if, after uniformly changing every product type $\mathsf{E} = \langle E, B, W, \{\psi_e\}_{e \in E}\rangle$ appearing in the syntax to $F\mathsf{E} = \langle E, FB, W, \{\psi_e\}_{e \in E}\rangle$, the resulting interpretation remains unchanged under transformation of models induced by F.[7] We then have the following result:

Theorem 5.3. *Let $F : \mathscr{A} \to \mathscr{B}$ be a modal functor between two modal categories, and suppose the semantic functor $(-)^+_{\mathscr{B}}$ is injective on objects. F further preserves the interpretation of $\mathscr{L}^{\mathbf{PRO}}$ iff it preserves pullback maps and fibre-wise finite meets.*

Proof. The if part can be proven by a straight forward induction on the complexity of formulas in $\mathscr{L}^{\mathbf{PRO}}$. The only if part is technically trickier, though the general idea is no different from previous proofs of such correspondence results. We include a detailed proof in Appendix A for the convenience of referees. $\square$

6   Conclusion

In this paper, we have used the language of topological categories to provide a unifying description of different types of semantics of modal logic, and have showed how various semantic structures within topological categories enable us to interpret different extensions of modal logic, including modal strength, group structure, and logical dynamics. We believe our approach is instructive for the current active research in the modal logic world on related topics.

For each fragment we have also proven a correspondence result, showing the equivalence for a concrete functor to preserve the interpretation of that fragment and for it to preserve certain categorical structures. Such results have established a close connection between the syntax and semantics of modal logic, and have deepened our understanding of its abstract mathematical structures. They can be seen as justification that topological category is a particularly nice framework to explore its further connections with modal logic.

Acknowledgement

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[7]A far more general approach is to look at the relationship between a model transformation induced by a concrete functor F, and a particular syntactic translation T. Our notion of preservation of languages is then a special case when T is the identity translation, or in the case of product type updates, translating E to FE. We leave this for future investigation.

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A   Proof of Theorem 5.3

To complete the other half of the proof, we roughly need to show that any initial lift and any finite meets in the fibre could be represented by some product type update with a specific chosen product type. First of all, since empty product type update is included in our dynamic extension $\mathscr{L}^{\mathbf{PRO}}$, to preserve it we may assume F already preserves the top element within each fibre. We first show that F commutes with pullback maps. Suppose for some function $\pi : E \to X$, F does not commute with the initial lifts on π in $\mathscr{A}$ and $\mathscr{B}$. This means that we have some object A in the fibre $\mathscr{A}_X$, such that $F\pi^*A$ and π^*FA are two distinct objects in $\mathscr{B}_X$. Now since the semantic functor on $\mathscr{B}$ is injective on objects, the induced operators $(\pi^*FA)_{\mathscr{B}}^+$, which we denote as m, and $(F\pi^*A)_{\mathscr{B}}^+$, which we denote as m', will be distinct, which means they disagree on some subset T of E.

Now consider the product type $\mathsf{E} = \langle E, \top_E^{\mathscr{A}}, W, \{p_e\}_{e \in E} \rangle$, where the family of formulas is an E-indexed family of distinct propositional letters. For the evaluation function W on E, we require that for some propositional letter q distinct from p_e for any $e \in E$, we have $W(q) = T$. Now consider an evaluation function V on X, such that for any $e \in E$ we have $V(p_e) = \{\pi(e)\}$, which means that $[\![p_e]\!]_A^V$ is a singleton for any $e \in E$. We also requires that $V(q) = X$. Then by definition, we have

$$E \otimes_V X = \sum_{e \in E} [\![p_e]\!]_A^V = E,$$

and it is not hard to see that the projection map π_E is the identity on E, and π_X is simply given by π. Notice that, the above statement of the underlying set of the updated model remains true even if we have calculated it in $\mathscr{B}$.

Now by definition, the geometric data on the updated model is calculated as follows,

$$\mathsf{E} \otimes_V A = 1_E^* \top_E^{\mathscr{A}} \wedge \pi^*A = \pi^*A,$$

and for the induced product update in $\mathscr{B}$,

$$F\mathsf{E} \otimes_V FA = 1_E^* F\top_E^{\mathscr{A}} \wedge \pi^*FA = \top_E^{\mathscr{B}} \wedge \pi^*FA = \pi^*FA.$$

The above uses the fact that initial lifts preserves top elements since it is a right adjoint, and the assumption that F preserves top elements in the fibres. As for the evaluation function $W \otimes V$, it is easy to calculate that

$$(W \otimes V)(q) = W(q) \wedge \pi^{-1}V(q) = W(q) = T.$$

Finally, consider the interpretation of the formula $\langle \mathsf{E}, \{e\} \rangle \square q$, where e is some element in E such that $e \in m(T)$ but $e \notin m'(T)$ (or the other way around). Then by definition, we have the following calculation,

$$[\![\langle \mathsf{E}, \{e\} \rangle \square q]\!]_A^V = \exists_\pi(\{(e, \pi(e))\} \cap [\![\square q]\!]_{\pi^*A}^{W \otimes V}) = \exists_\pi(\{(e, \pi(e))\} \cap [\![\square q]\!]_{F\pi^*A}^{W \otimes V}) = \emptyset.$$

The first equality is due to the fact that $E \otimes_V A = \pi^*A$ as we have shown above; the second equality is by the fact that F preserves the interpretation of $\mathscr{L}$; and the final equality holds because we have assumed $e \notin m'(T)$. On the other hand, we have the other calculation as follows,

$$[\![\langle F\mathsf{E}, e \rangle \square q]\!]_{FA}^V = \exists_\pi(\{(e, \pi(e))\} \cap [\![\square q]\!]_{F\mathsf{E} \otimes_V FA}^{W \otimes V}) = \exists_\pi(\{(e, \pi(e))\} \cap [\![\square q]\!]_{\pi^*FA}^{W \otimes V}) = \{\pi(e)\}.$$

These calculation are basically the same as before, only that in the final step, the result is a singleton $\{\pi(e)\}$ because $e \in m(T)$. This constructions shows that F would then not preserve the interpretation of

the formula $\langle \mathsf{E}, \{e\} \rangle \Box q$ on this particular model. Hence, F must preserves the initial lift of any single structured sources.

Furthermore, we need to show that F preserves the binary meets fibre-wise as well. The basic idea is the same. Suppose F does not preserve binary meets in the fibre, then for some set X and some A, B in the fibre $\mathscr{A}_X$, we would have $F(A \wedge B)$ distinct from $FA \wedge FB$ in $\mathscr{B}_X$. Again, the operators m, m' associated to $F(A \wedge B)$ and $FA \wedge FB$ would differ on some subset T of X; we let $y \in X$ be the element in $m(T)$ but not $m'(T)$ (or the other way around). We can then construct the product type X as follows,

$$\mathsf{X} = \langle X, B, \top_X^{\mathbf{Evl}}, \{p_x\}_{x \in X} \rangle.$$

We also consider the model A on X, with a chosen evaluation function V satisfying the following condition: For any $x \in X$, we have $V(p_x) = \{x\}$, and for another distinct variable q we have $V(q) = T$. The product type update would result in the following model,

$$\mathsf{X} \otimes_V \mathsf{X} = \sum_{x \in X} [\![p_x]\!]_A^V = X,$$

and the two projection maps are both the identity function 1_X on X. Again, this is independent of the modal categories $\mathscr{A}$ or $\mathscr{B}$. The topology categorical structure on the updated model, calculated in $\mathscr{A}$, is simply given as follows,

$$\mathsf{X} \otimes_V A = 1_X^* B \wedge 1_X^* A = A \wedge B.$$

In the modal category $\mathscr{B}$ however, we have

$$F\mathsf{X} \otimes_V FA = 1^* XFB \wedge 1^* XFA = FA \wedge FB.$$

In both cases, it is easy to see that the updated evaluation function $\top_X^{\mathbf{Evl}} \otimes V$ remains to be V itself.

By definition, consider the evaluation of the formula $\langle \mathsf{X}, \{y\} \rangle \Box q$. On one hand,

$$[\![\langle \mathsf{X}, \{y\} \rangle \Box q]\!]_A^V = \{y\} \cap [\![\Box q]\!]_{A \wedge B}^V = \{y\} \cap [\![\Box q]\!]_{F(A \wedge B)}^V = \{y\} \cap m(T) = \{y\}.$$

On the other hand,

$$[\![\langle F\mathsf{X}, \{y\} \rangle \Box q]\!]_{FA}^V = \{y\} \cap [\![\Box q]\!]_{FA \wedge FB}^V = \{y\} \cap m'(T) = \emptyset.$$

Hence, this explicitly constructs a formula where F does not preserve its interpretation, and this completes the proof.