Morphisms in categories of nonlocal games

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Overview

We study *correlations*, conditional probability densities on *finite* sets

$$p(y_A, y_B \mid x_A, x_B) \text{ where } x_A, x_B \in X \text{ and } y_A, y_B \in Y.$$

A *local*, or *hidden variables*, or simply *classical*, correlation takes the form

$$p(y_A, y_B \mid x_A, x_B) = \sum_{\omega\in\Omega} \mu(\omega)\mathrm{Pr}_A(y_A \mid x_A, \omega)\mathrm{Pr}_B(y_B \mid x_B, \omega).$$

A *quantum* correlation takes the form

$$p(y_A, y_B \mid x_A, x_B) = \mathrm{tr}(\rho(E^{x_A}_{y_A} \otimes F^{x_B}_{y_B}))$$

In general, a *nonsignaling* correlation satisfies the conditions:

$$\textstyle\sum_{y_B} p(y_A, y_B \mid x_A, x_B) = \sum_{y_B} p(y_A, y_B \mid x_A, x'_B) \text{ for all } y_A, x_A, x_B, x'_B,$$
$$\textstyle\sum_{y_A} p(y_A, y_B \mid x_A, x_B) = \sum_{y_A} p(y_A, y_B \mid x'_A, x_B) \text{ for all } y_B, x_A, x_B, x'_A.$$

A correlation is *synchronous* if it satisfies:

$$p(y_A, y_B \mid x, x) = 0 \text{ whenever } x \in X \text{ and } y_A \neq y_B \text{ in } Y.$$

Lemma. Under natural composition of probability densities:

- the composition of synchronous correlations is synchronous;
- the composition of nonsignaling correlations is nonsignaling;
- the composition of quantum correlations is quantum;
- the composition of classical correlations is classical.

Consequence. We can construct categories each of whose *objects* are finite sets, and whose *morphisms* are the nonlocal games with synchronous correlations (FinSet^S), or synchronous classical (FinSet^S_{HV}), synchronous quantum (FinSet^S_Q), or synchronous nonsignalling (FinSet^S_{NS}) correlations.

Main result. We classify the categorical notions of one-to-one (section and monomorphism) and onto (retraction and epimorphism) in each category. Unfortunately these conditions cannot be used to separate hidden variables from quantum (or general nonsignalling) correlations.

Sections and Monomorphisms

A morphism $\alpha \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is a *section* if there is a $\beta \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ with $\beta \circ \alpha = \mathrm{id}_A$. It is a *monomorphism* if whenever $\gamma_1, \gamma_2 \in \mathrm{Hom}_{\mathsf{C}}(Z, A)$ have $\alpha \circ \gamma_1 = \alpha \circ \gamma_2$ then $\gamma_1 = \gamma_2$. In FinSet both mean α is one-to-one.

Theorem 1. In FinSet^S the sections are precisely the correlations

$$p(y_A, y_B|x_A, x_B) = \mathbb{1}_{\{y_A = f_A(x_A,x_B)\}}\mathbb{1}_{\{y_B = f_B(x_A,x_B)\}}$$

such that (i) $(f_A, f_B) : X^2 \to Y^2$ is one-to-one and (ii) $f_A(x_A, x_B) = f_B(x_A, x_B)$ if and only if $x_A = x_B$.

Corollary. The sections in FinSet^S_{HV}, FinSet^S_Q, and FinSet^S_{NS} are precisely the sections in FinSet (i.e. deterministic correlations from one-to-one functions).

Theorem 2. In each of these categories, the monomorphisms are precisely those correlations whose stochastic matrices have zero right nullspace.

Theorem 1 is easy to prove. Theorem 2 is more challenging:

- it is straightforward to prove for FinSet^S;
- for FinSet^S_{NS} on needs Lemma 1 and the following lemma;

Lemma. Let p be a nonsignaling synchronous correlation, P its stochastic matrix, and suppose $P\mathbf{u} = \mathbf{0}$. Define $w_1(x_A, x_B) = \mathbb{1}_{\{x_A=x_B\}} \sum_z u(x_A, z)$ and $w_2(x_A, x_B) = \mathbb{1}_{\{x_A=x_B\}} \sum_z u(z, x_B)$. Then we have $P\mathbf{w}_1 = P\mathbf{w}_2 = \mathbf{0}$.

- for FinSet^S_{HV} and FinSet^S_Q on uses Lemma 2 and the following result.

Lemma. Let $p \in \mathrm{Hom}^S(X, Y)$ be a symmetric with associated stochastic matrix P. Suppose $P\mathbf{u} = \mathbf{0}$. Then $v(x_A, x_B) = u(x_B, x_A)$ also has $P\mathbf{v} = \mathbf{0}$.

Retractions and Epimorphisms

A morphism $\alpha \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is a *retraction* if there is a $\beta \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ with $\alpha \circ \beta = \mathrm{id}_B$. It is an *epimorphism* if whenever $\gamma_1, \gamma_2 \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ have $\gamma_1 \circ \alpha = \gamma_2 \circ \alpha$ then $\gamma_1 = \gamma_2$. In FinSet both mean α is onto.

Theorem 3. In FinSet^S the retractions are precisely the correlations

$$p(y_A, y_B|x_A, x_B) = \mathbb{1}_{\{y_A = f_A(x_A,x_B)\}}\mathbb{1}_{\{y_B = f_B(x_A,x_B)\}}$$

such that (i) $F = (f_A, f_B) : X^2 \to Y^2$ is onto, (ii) $f_A(x, x) = f_B(x, x)$, and (iii) for each $y \in Y$ we have $f_A(x, x) = y = f_B(x, x)$ for some $x \in X$.

Corollary. The retractions in FinSet^S_{HV}, FinSet^S_Q, and FinSet_{NS} are precisely the retractions in FinSet (i.e. deterministic correlations from onto functions).

Theorem 4. In each category, the epimorphisms are precisely those correlations whose stochastic matrices have zero left nullspace.

Theorem 3 is dual to Theorem 1, however Theorem 4 needs a new proof:

- again the proof is simple for FinSet^S;
- the proof for FinSet^S_{NS} is more challenging but still direct;
- the proofs for FinSet^S_{HV} and FinSet^S_Q rely on
  - 1. synchronous classical and quantum correlations are symmetric [1, Corollary 11],
  - 2. applying the lemma below to reduce to the symmetric and antisymmetric kernels,
  - 3. directly proving an analogous result for each of these.

Lemma. Let p be a symmetric synchronous correlation with associated stochastic matrix P. Suppose $\underline{w}P = \underline{0}$. Then $v(y_A, y_B) = w(y_B, y_A)$ also has $\underline{v}P = \underline{0}$. In particular, if we decompose into its symmetric and antisymmetric parts of $\underline{w}$ have $\underline{w}^{(\pm)}P = \underline{0}$.

Isomorphism versus Bimorphism

An *isomorphism* is a both a section and a retractions, so in FinSet^S_{HV}, FinSet^S_Q, and FinSet^S_{NS} these are just bijective functions. A *bimorphism*, which is morphism that is both epic and monic, is strictly weaker: a correlation in any of these categories is a bimorphism when its stochastic matrix is nonsingular.

Consequently, finite sets equivalent under quantum bimorphism still have the same cardinality, and so are isomorphic in the usual sense.

Technical Lemmas

Lemma 1. ([1]) Let Y be a finite set and $u = u(y_A, y_B)$ and $v = v(y_A, y_B)$ be probability distributions on Y^2 such that for all $y \in Y$:

$$\sum_{y'} u(y, y') = \sum_{y'} v(y', y) =: \theta(y) \text{ and } \sum_{y'} u(y', y) = \sum_{y'} v(y, y') =: \phi(y).$$

Then the following defines a synchronous nonsignaling correlation:

$$p(y_A, y_B \mid 0, 0) = \mathbb{1}_{\{y_A=y_B\}}\theta(y_A) \qquad p(y_A, y_B \mid 0, 1) = u(y_A, y_B)$$
$$p(y_A, y_B \mid 1, 0) = v(y_A, y_B) \qquad p(y_A, y_B \mid 1, 1) = \mathbb{1}_{\{y_A=y_B\}}\phi(y_A).$$

Also, every synchronous nonsignaling correlation from $\{0, 1\}$ to Y arises this way.

Lemma 2. ([1]) Let Y be a finite set and $u = u(y_A, y_B)$ be a probability distribution on Y^2. Write $\theta(y) = \sum_{y'} u(y, y')$ and $\phi(y) = \sum_{y'} u(y', y)$. Define

$$p(y_A, y_B \mid 0, 0) = \mathbb{1}_{\{y_A=y_B\}}\theta(y_A) \qquad p(y_A, y_B \mid 0, 1) = u(y_A, y_B)$$
$$p(y_A, y_B \mid 1, 0) = u(y_B, y_A) \qquad p(y_A, y_B \mid 1, 1) = \mathbb{1}_{\{y_A=y_B\}}\phi(y_A).$$

Then p is a synchronous classical correlation. Also, every synchronous classical correlation from $\{0, 1\}$ to Y arises this way.

Corollary. Every symmetric synchronous nonsignaling correlation with domain $\{0, 1\}$ is classical. In particular, any synchronous quantum correlation with two measurement settings is classical.

Lemma 3. ([1]) Suppose $|X| \geq 2$ and let $w = w(x_A, x_B)$ be a nonnegative function on X^2 such that for every $x_A, x_B \in X$:

$$w(x_A, x_B) \leq w(x_A, x_A),\ w(x_A, x_B) \leq w(x_B, x_B), \text{ and}$$
$$w(x_A, x_A) + w(x_B, x_B) \leq 1 + w(x_A, x_B).$$

Then the following defines a synchronous nonsignaling correlation:

$$p(0, 0 \mid x_A, x_B) = 1 + w(x_A, x_B) - w(x_A, x_A) - w(x_B, x_B)$$
$$p(0, 1 \mid x_A, x_B) = w(x_B, x_B) - w(x_A, x_B)$$
$$p(1, 0 \mid x_A, x_B) = w(x_A, x_A) - w(x_A, x_B)$$
$$p(1, 1 \mid x_A, x_B) = w(x_A, x_B).$$

Also, every synchronous nonsignaling correlation from X to $\{0, 1\}$ arises in this way.

[1] Lackey, Rodrigues, "Nonlocal games, synchronous correlations, and Bell inequalities," arXiv:1707.06200