

COLLECTIVES: COMPOSITIONAL PROTOCOLS FOR CONTRIBUTIONS AND RETURNS

Nelson Niu (Univ. of Washington) & David I. Spivak (Topos Institute)

Definition: What is a collective?

A **collective** is a set C of **contributions** equipped with an associative **aggregation** operation $*: C \times C \rightarrow C$ with unit $1 \in C$ —so $(C, 1, *)$ is a monoid, modeling a *mutual endeavor*—and a set $R[c]$ of **returns** on each contribution $c \in C$ equipped with a **distribution** operation

$$\left(\frac{a}{a * b}, \frac{b}{a * b} \right) : R[a * b] \rightarrow R[a] \times R[b]$$

for each $a, b \in C$ —modeling *division of returns*—such that

$$\frac{a}{a * 1} = \text{id}_{R[a]} = \frac{a}{1 * a}$$

and the following coassociativity square commutes:

$$\begin{array}{ccc} R[a * b * c] & \xrightarrow{\left(\frac{a * b}{(a * b) * c}, \frac{c}{(a * b) * c} \right)} & R[a * b] \times R[c] \\ \downarrow \left(\frac{a}{a * (b * c)}, \frac{b * c}{a * (b * c)} \right) & & \downarrow \left(\frac{a}{a * b}, \frac{b}{a * b} \right) \times R[c] \\ R[a] \times R[b * c] & \xrightarrow{R[a] \times \left(\frac{b}{b * c}, \frac{c}{b * c} \right)} & R[a] \times R[b] \times R[c]. \end{array}$$

Collectives = monoids in $(\mathbf{Poly}, y, \otimes)$

A **collective** with **contributions** C and **returns** $(R[c])_{c \in C}$ is a monoid in the category of polynomial functors carried by

$$\sum_{c \in C} y^{R[c]},$$

with respect to the monoidal **parallel product** \otimes given by

$$\sum_{i \in I} y^{X_i} \otimes \sum_{j \in J} y^{Y_j} := \sum_{i \in I} \sum_{j \in J} y^{X_i \times Y_j}.$$

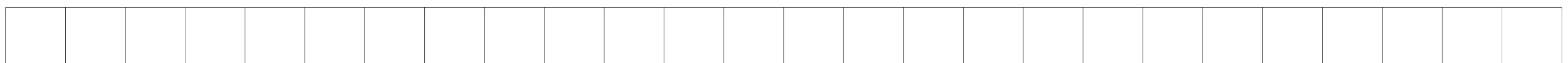
There is a **free collective** on any polynomial $p := \sum_{i \in I} y^{X_i}$ carried by

$$\sum_{n \in \mathbb{N}} p^{\otimes n} = \sum_{n \in \mathbb{N}} \sum_{i_1, \dots, i_n \in I} y^{X_{i_1} \times \dots \times X_{i_n}}.$$

Given collectives carried by $p, q \in \mathbf{Poly}$, there is a collective carried by:

- $p \otimes q$, putting the collectives in **parallel**
- $p \circ q$, putting the collectives in **series**—as **Poly** has a duoidal structure $(p \circ q) \otimes (p' \circ q') \rightarrow (p \otimes p') \circ (q \otimes q')$

Example: Make a contribution!



Example: Potluck

Contribution: subset $V \subseteq U$, where V is a set of *dishes offered* from some universe of dishes U

Return: subset $X \subseteq V$, where X is a set of *dishes requested* among those offered in V

Aggregation: $V * W = V \cup W$

Distribution: requests passed down, with repeats passed to the left but not to the right—so request $Y \subseteq V \cup W$ distributes as

$$V \cap Y, \quad (W \cap Y) \setminus V$$

Example: Prediction market

Contribution: (k, p) , where $k \in \mathbb{N}$ is the *number of bettors* and $p: \Omega \rightarrow [0, 1]$ are their (normalized) *bets* on a sample space Ω

Return: (o, r) , where $o \in \Omega$ is the *observed outcome* and $r \in \mathbb{R}_{\geq 0}$ is the *reward*

Aggregation: $(k, p) * (\ell, q) = \left(k + \ell, \frac{kp + \ell q}{k + \ell} \right)$, a *weighted average*

Distribution: outcome announced, reward proportional to bets—so (o, r) distributes as

$$\left(o, \frac{kp(o)}{kp(o) + \ell q(o)} \cdot r \right), \left(s, \frac{\ell q(o)}{kp(o) + \ell q(o)} \cdot r \right)$$

Example: Survey

Contribution: $m \in \mathbb{N}$, encoding a *multiple-choice question* with m possible answer choices

Return: $a \in \{0, \dots, m-1\}$, encoding a chosen answer

Aggregation: $m * n = mn$

Distribution: answer to mn -choice question interpreted as answer to m -choice question and n -choice question, by distributing $b \in \{0, \dots, mn-1\}$ as

$$a \bmod m, \quad a \div m$$

Acknowledgments

We thank Christian Williams for insightful discussions and suggestions. The prediction market example is due to Spencer Breiner. This material is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-20-1-0348.