COLLECTIVES: COMPOSITIONAL PROTOCOLS FOR CONTRIBUTIONS AND RETURNS

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Definition: What is a collective?

A ***collective*** is a set C of **contributions** equipped with an associative **aggregation** operation $*\colon C \times C \to C$ with unit $1 \in C$—so $(C, 1, *)$ is a monoid, modeling a *mutual endeavor*—and a set $R[c]$ of **returns** on each contribution $c \in C$ equipped with a **distribution** operation

$$\left(\frac{a}{a*b}, \frac{b}{a*b}\right) : R[a*b] \to R[a] \times R[b]$$

for each $a, b \in C$—modeling *division of returns*—such that

$$\frac{a}{a*1} = \mathrm{id}_{R[a]} = \frac{a}{1*a}$$

and the following coassociativity square commutes:

$$\begin{array}{ccc}
R[a*b*c] & \xrightarrow{\left(\frac{a*b}{(a*b)*c}, \frac{c}{(a*b)*c}\right)} & R[a*b] \times R[c] \\
{\scriptstyle \left(\frac{a}{a*(b*c)}, \frac{b*c}{a*(b*c)}\right)}\big\downarrow & & \big\downarrow{\scriptstyle \left(\frac{a}{a*b}, \frac{b}{a*b}\right)\times R[c]} \\
R[a] \times R[b*c] & \xrightarrow[R[a]\times\left(\frac{b}{b*c}, \frac{c}{b*c}\right)]{} & R[a] \times R[b] \times R[c].
\end{array}$$

Collectives = monoids in $(\mathbf{Poly}, y, \otimes)$

A ***collective*** with **contributions** C and **returns** $(R[c])_{c\in C}$ is a monoid in the category of polynomial functors carried by

$$\sum_{c\in C} y^{R[c]},$$

with respect to the monoidal **parallel product** $\otimes$ given by

$$\sum_{i\in I} y^{X_i} \otimes \sum_{j\in J} y^{Y_j} := \sum_{i\in I}\sum_{j\in J} y^{X_i \times Y_j}.$$

There is a ***free collective*** on any polynomial $p := \sum_{i\in I} y^{X_i}$ carried by

$$\sum_{n\in\mathbb{N}} p^{\otimes n} = \sum_{n\in\mathbb{N}} \sum_{i_1,\ldots,i_n\in I} y^{X_{i_1}\times\cdots\times X_{i_n}}.$$

Given collectives carried by $p, q \in \mathbf{Poly}$, there is a collective carried by:

- $p \otimes q$, putting the collectives in ***parallel***
- $p \circ q$, putting the collectives in ***series***—as $\mathbf{Poly}$ has a duoidal structure $(p \circ q) \otimes (p' \circ q') \to (p \otimes p') \circ (q \otimes q')$

Example: Make a contribution!

Example: Potluck

Contribution: subset $V \subseteq U$, where V is a set of *dishes offered* from some universe of dishes U
Return: subset $X \subseteq V$, where X is a set of *dishes requested* among those offered in V
Aggregation: $V * W = V \cup W$
Distribution: requests passed down, with repeats passed to the left but not to the right—so request $Y \subseteq V \cup W$ distributes as

$$V \cap Y, \quad (W \cap Y) \setminus V$$

Example: Prediction market

Contribution: (k, p), where $k \in \mathbb{N}$ is the *number of bettors* and $p\colon \Omega \to [0,1]$ are their (normalized) *bets* on a sample space Ω
Return: (o, r), where $o \in \Omega$ is the *observed outcome* and $r \in \mathbb{R}_{\geq 0}$ is the *reward*
Aggregation: $(k,p) * (\ell, q) = \left(k + \ell, \frac{kp+\ell q}{k+\ell}\right)$, a *weighted average*
Distribution: outcome announced, reward proportional to bets—so (o, r) distributes as

$$\left(o, \frac{kp(o)}{kp(o)+\ell q(o)} \cdot r\right), \left(s, \frac{\ell q(o)}{kp(o)+\ell q(o)} \cdot r\right)$$

Example: Survey

Contribution: $m \in \mathbb{N}$, encoding a *multiple-choice question* with m possible answer choices
Return: $a \in \{0, \ldots, m-1\}$, encoding a chosen answer
Aggregation: $m * n = mn$
Distribution: answer to mn-choice question interpreted as answer to m-choice question and n-choice question, by distributing $b \in \{0, \ldots, mn-1\}$ as

$$a \bmod m, \quad a \operatorname{div} m$$

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