A Framework for Universality Across Disciplines

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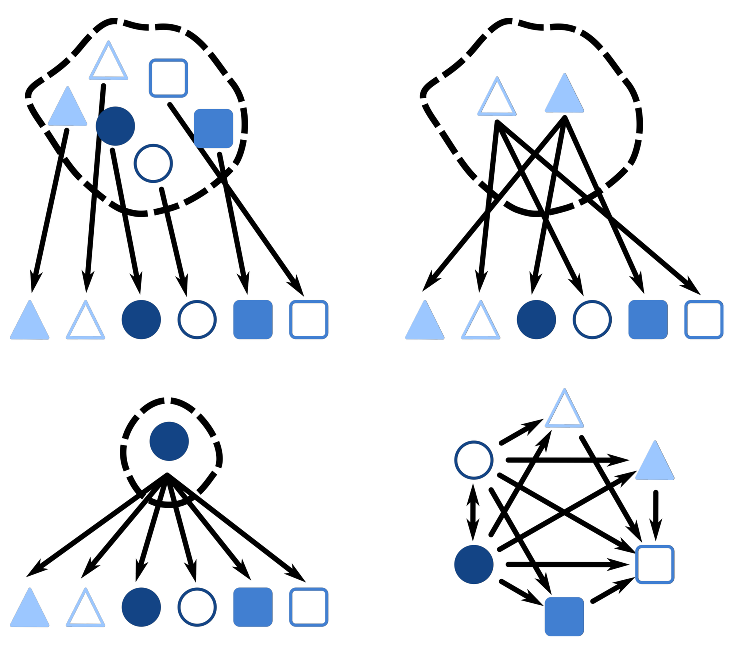
Concept

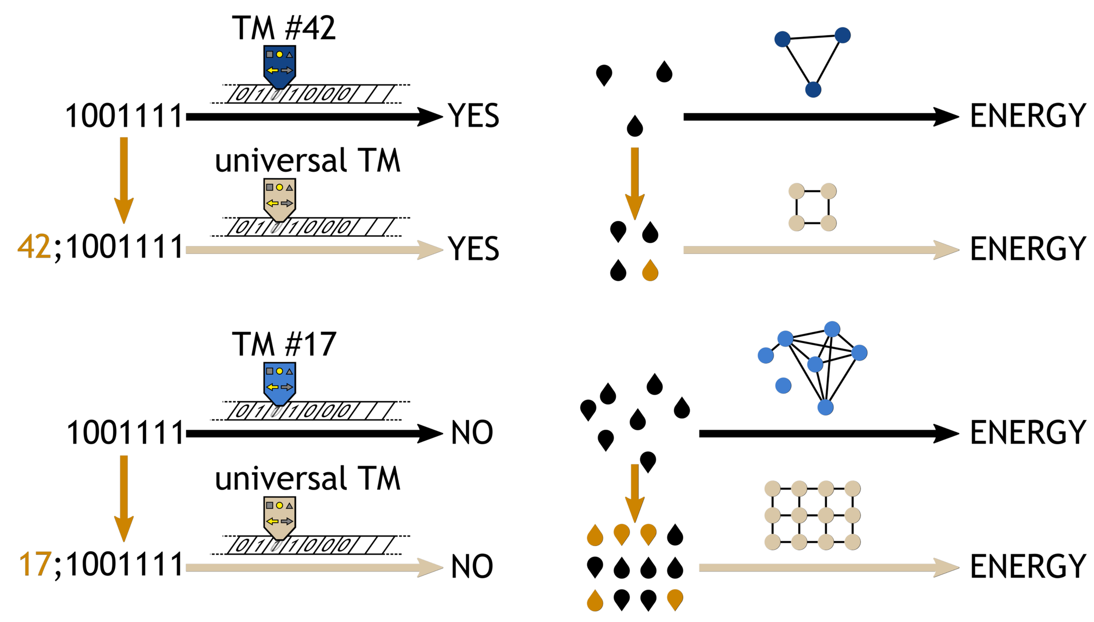
Universality is a concept appearing in many different fields, with varying meaning. There are universal gate sets, universal Turing machines, universal spin models [1], and many more. We develop a framework to compare different instances of universality and to study consequences of universality, such as undecidability.



Most instances of universality can be understood as universal simulation; we consider a set of transformations taking an input to an output. Simulating a transformation then means reproducing the corresponding output for every input. A transformation or a subset of transformations is universal if it can simulate all other transformations in a given domain. The drawing shows a selection of instances of universality.

Instances

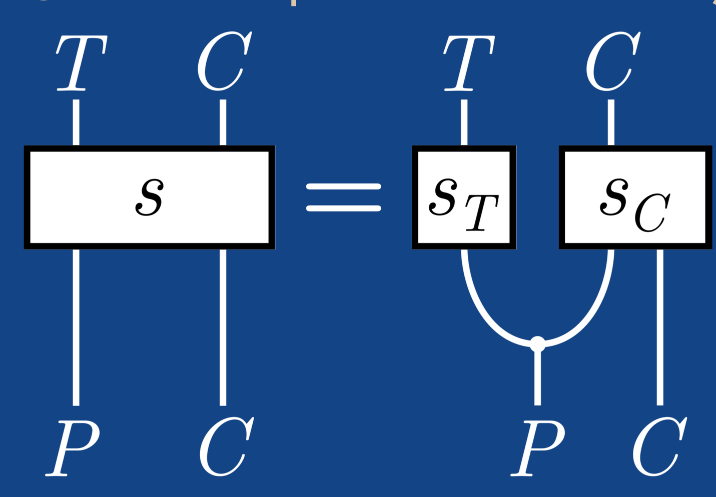
Turing machines (TMs) and spin models are prominent examples of universality. The drawing below shows how a universal Turing machine simulates other Turing machines and how the 2D Ising model with fields (a universal spin model) simulates other spin models.



In both cases, the simulation requires a modification of the input. For Turing machines a single universal machine (a single transformation) is sufficient, however, spin model universality also requires a choice of a transformation.

Simulator

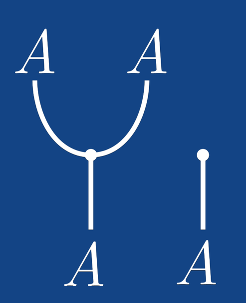
The main object of our study are simulators, morphisms of type $s: P \otimes C \to T \otimes C$ which split in the following way:



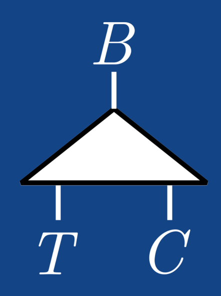
where we think of P as programs or descriptions for transformations. A simulator captures both a modification of the input and a choice of transformation. The splitting ensures that the chosen transformation is independent of the input.

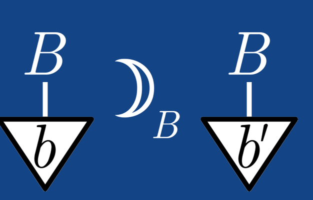
Basic Set-Up

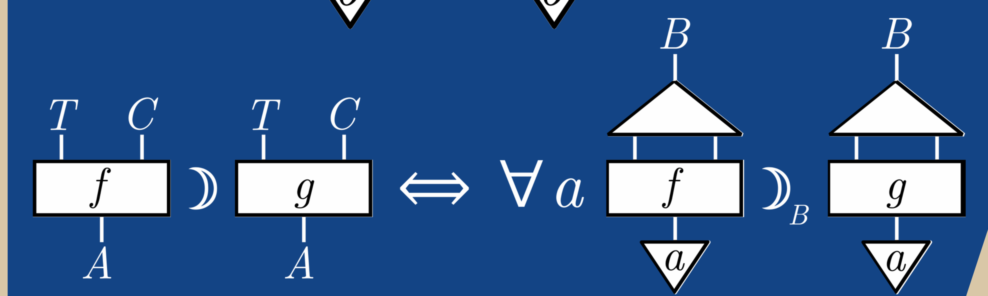
The framework is built on gs-monoidal categories [5], symmetric monoidal categories with a notion of copying and deleting for each object.



The main ingredient is a morphism $eval: T \otimes C \to B$, where T are some kind of transformations, C are contexts on which we want to compare or evaluate the transformations and B are behaviors they exhibit when evaluated. Finally, we relax equality by introducing a relation $\succ_B$ on generalized elements of B and extending it to morphisms $A \to T \otimes C$.

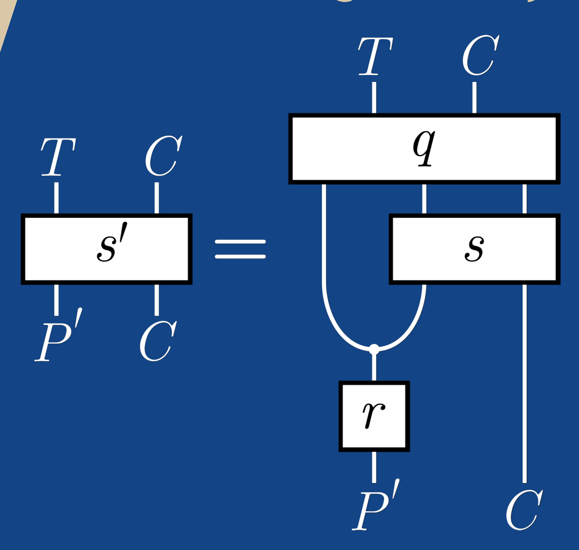


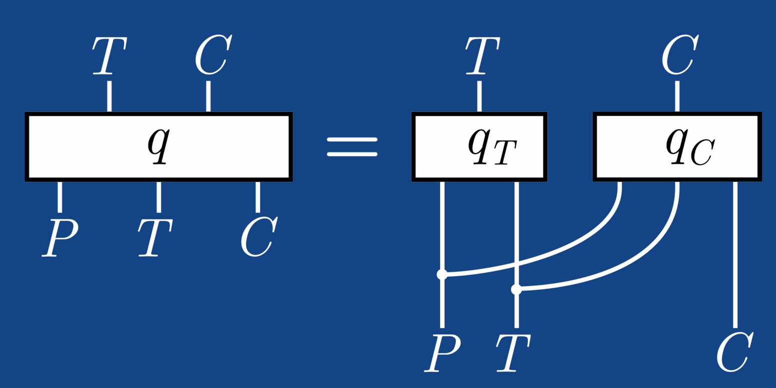


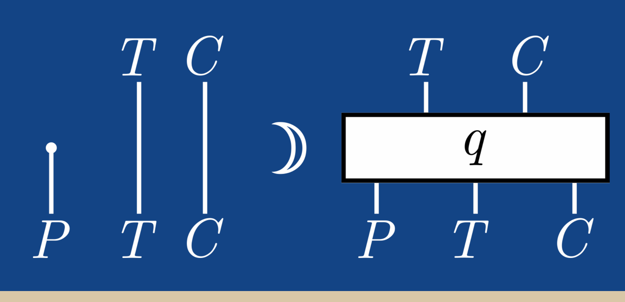


Morphisms of Simulators

To further investigate simulators, e.g. to distinguish trivial from non-trivial simulators, we consider the category where the objects are simulators and morphisms $s \to s'$ are given by tuples (r, q), where $r: P' \to P$ is called a reduction and $q: P \otimes T \otimes C \to T \otimes C$ is called a post-processing, satisfying the following diagrams. Morphisms of this kind reflect universality.

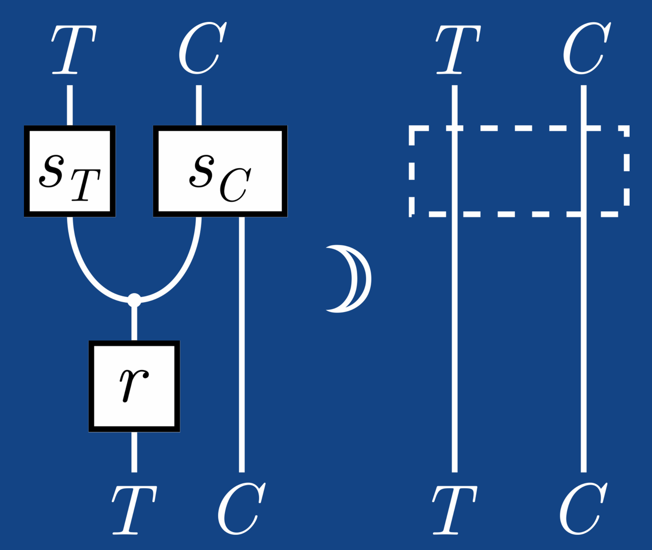






Universality

Intuitively, a simulator is universal if there is a description for any transformation for which the simulator simulates said transformation. Formally, we use a reduction $r: T \to P$ and define: A simulator s is universal if:



Where the right hand side is the trivial universal simulator, which always exists.

Examples

For Turing machines, Turing categories [6] fit the framework: T is a Turing object, $C = B = \Sigma^*$ are strings and $eval$ evaluates Turing machines on strings. For spin models, C are spin configurations, T are spin models and $eval$ maps them to energies. In both cases, the usual notions of universality can be seen as universal simulators.

Undecidability

Universality and undecidability occur jointly in some instances, the most prominent example is the Halting problem for Turing machines. Undecidability is usually proven with diagonalization arguments, which are instances of Lawvere's theorem [2]. Our framework provides conditions for universality to imply undecidability.

Outlook

This framework is a first step in gaining a better understanding of universality and its consequences. Many questions remain open, for example:

- What are other meaningful instances of universality?
- How can we compare instances of universality? When are their universalities equivalent?
- Some universal simulators are trivial, others are not. Under which conditions does non-trivial universality occur? And can we classify universal simulators by their strengths ?
- There are other consequences of universality, such as the jump to universality [4]. How can they be expressed in our framework?

References

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