Categories of Kirchhoff Relations

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Introduction

- The goal of this work was to build a bridge between electrical circuits and quantum circuits.
- The connection to electrical circuits is made, using subcategories of Lagrangian relations that satisfy Kirchhoff current law.
- Comfort and Kissinger showed that $\mathsf{AffLagRel}_F$ is isomorphic to qudit stabilizer circuits. [1]

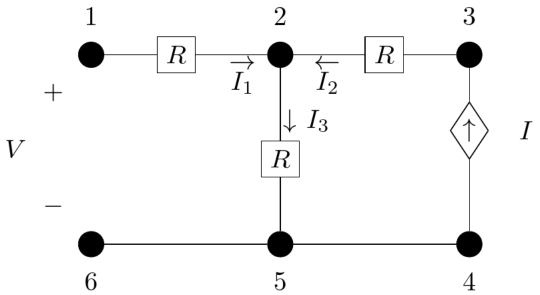
$$\mathsf{ResRel}_F \hookrightarrow \mathsf{KirRel}_F \hookrightarrow \mathsf{AffLagRel}_F = \begin{array}{c}\text{Stabilizer}\\ \text{QM}\\ \text{(odd-prime)}\end{array}$$

[1] Cole Comfort, and Aleks Kissinger. "A Graphical Calculus for Lagrangian Relations." arXiv preprint arXiv:2105.06244 (2021).

Main modes of attack

- **Graphical Calculus**: reasoning about relations using pictures!
- **Parity-Check Matrices**: Reasoning about relations using matrices.

Basics of Electrical Circuits

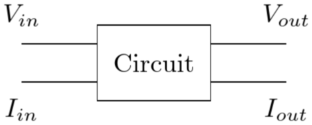


- **Kirchhoff's current law**: (KCL) is just a restatement of conservation of charge, for example at node 2, KCL will give:

$$I_3 = I_1 + I_2$$

- **Kirchhoff's voltage law**: (KVL) states that the voltage drop in a loop is zero.

Circuits as relations



- A circuit can be viewed as a **relation** between input and output currents and voltages.
- One can use a matrix to capture this relationship:

$$Hx = 0$$

where H is a matrix which characterizes the circuit and

$$x = \begin{pmatrix} V_{in} & V_{out} & I_{in} & I_{out} \end{pmatrix}^T$$

Category of Linear Relations

The category of **linear relations** LinRel_F over a field F is defined as:

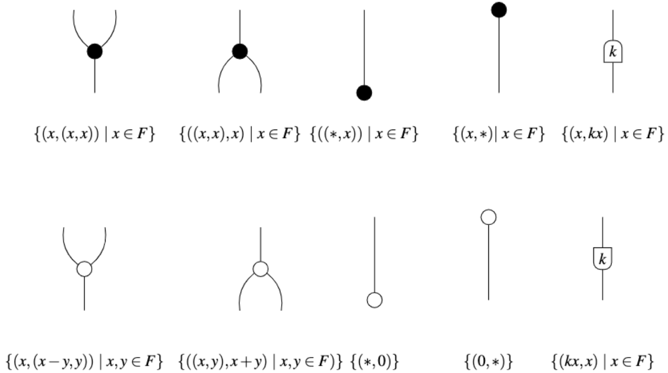
- Objects: Natural Numbers, $m, n \in \mathbb{N}$
- Maps: $\mathcal{R} : m \to n$ where $\mathcal{R} \subseteq F^m \times F^n$. The morphisms are linear subspaces.
- Composition: Given two relations $\mathcal{R}_1 : m \to n$ and $\mathcal{R}_2 : n \to p$, the composite relation $\mathcal{R}_2 \circ \mathcal{R}_1 : m \to p$ is:

$$\mathcal{R}_2 \circ \mathcal{R}_1 := \{(x, z) \in F^m \times F^n : \ \exists y.\ (x, y) \in \mathcal{R}_1 \text{ and } (y, z) \in \mathcal{R}_2\}$$

- Identity: $\{(a, a) \mid a \in F^m\}$.

Category of Linear Relations

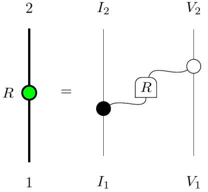
- The category LinRel_F has a graphical calculus whose generators are[2]:



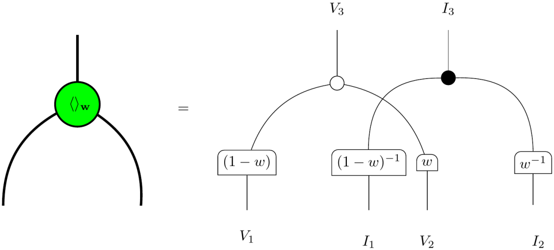
Read all these diagrams bottom up!

[2]Bonchi, F., Piedeleu, R., Sobociński, P., & Zanasi, F. (2019, June). Graphical affine algebra. In 2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS) (pp. 1-12). IEEE.

Electrical Elements in GLA



Resistor



Current Divider

The KCL and KVL equations for a resistor are:

$$V_2 = V_1 + I_1 R \quad I_1 = I_2$$

Symplectic Vector Spaces

- A **symplectic form** is a bilinear, alternating, non-degenerate map $\omega(\_,\_) : V \otimes V \to F$. A vector space with a symplectic form is called a **symplectic vector space**.
- A symplectic vector space always has a **Darboux basis** that is a basis $q_1, ..., q_n,\ p_1, ..., p_n$ satisfying $\omega(q_i, p_j) = -\omega(p_j, q_i) = \delta_{ij}$ and $\omega(q_i, q_j) = \omega(p_i, p_j) = 0$. Symplectic vector spaces are always even dimensional.
- In the Darboux basis, the symplectic form can be expressed using the $2n \times 2n$ block matrix J:

$$\omega(x, y) := \begin{pmatrix} q^T & p^T \end{pmatrix} J \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{where} \quad J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

Lagrangian Relations

- The **symplectic dual** of a linear subspace $U \subseteq V$ of a symplectic vector space V is the linear subspace $U^\omega := \{u' \in V \mid \forall u \in U,\ \omega(u', u) = 0\} \subseteq V$.
- A **Lagrangian subspace** U of a symplectic vector space V is linear subspace which is its own symplectic dual, so that $U = U^\omega$.

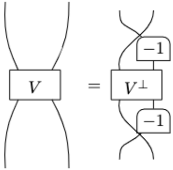
The Category of Lagrangian Relations

The category of Lagrangian relations over a field F consists of:

- Objects: $n \in \mathbb{N}$: correspond to the graded vector spaces $F^n \oplus F^n$ equipped with the canonical symplectic form given by J.
- Maps: $\mathcal{R} : n \to m$: are relations $\mathcal{R} \subseteq F^{n+m} \oplus F^{n+m}$ which is Lagrangian.
- Identity: Identity relation.
- Composition: Relational composition.

Lagrangian Relations: Graphical Definition

- A Lagrangian relation can be graphically be depicted as a linear relation V which satisfies the following condition [3]:



- $(-)^{\perp}$: $\mathsf{LinRel}_F \to \mathsf{LinRel}_F$, this maps a relation R to its orthogonal complement $R^{\perp}$.
- Using this definition one can show resistors, current dividers and junctions are Lagrangian relations!

[3] Comfort, Cole, and Aleks Kissinger. "A Graphical Calculus for Lagrangian Relations." arXiv preprint arXiv:2105.06244 (2021).

Parity-Check Matrices

- Any linear relation can be specified as a set of vectors $(x, y) \in F^m \oplus F^n$ satisfying a linear equation of the form:

$$H \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

  where $H : n + m \to n + m - k$ is a matrix of rank $n + m - k$. H is called a **parity check matrix**.

- H can be put using Gaussian elimination into a **standard form**:

$$H = \begin{pmatrix} 1_{m+n-k} & A \end{pmatrix} \sigma : m + n \to m + n - k,$$

  where σ is a permutation matrix.

Parity-Check Matrices for Lagrangian Relations

Theorem

The parity-check matrix for Lagrangian relations has the following standard form:

$$H = \begin{pmatrix} Y & 0 & 1_{n_p} & A^T \\ -A & 1_{n_q} & 0 & 0 \end{pmatrix} \sigma_S$$

where $n_p + n_q = n$ *(the dimension of* $\mathcal{R}$*),* σ_S *is a symplectic permutation,* A *has dimensions* $n_q \times n_p$*, and* $Y = Y^T$*.*

The Category of Kirchhoff Relations

A Lagrangian relation $\mathcal{R} : n \to m$ in LagRel_F satisfies:

1. **Kirchhoff's Current Law** if, for all $((q,p),(q',p')) \in \mathcal{R}$, the following equality holds:

$$\sum_{j=1}^{n} p_j = \sum_{k=1}^{m} p'_k$$

2. **Translation invariance** if, whenever $\lambda \in F$ and $((q,p),(q',p')) \in \mathcal{R}$, then $((q + \vec{\lambda}_m, p),(q' + \vec{\lambda}_n, p') \in \mathcal{R}$, where $\vec{\lambda}_n$ is a vector of dimension n all of whose components are the same $\lambda \in F$.

Lemma

For a state $\mathcal{R}$ in LagRel_F the following are equivalent:

1. *$\mathcal{R}$ satisfies the Kirchhoff current law;*
2. *$\mathcal{R}$ satisfies translational invariance.*

Parity-Check Matrices for Kirchhoff Relations

Theorem

The parity-check matrix for Kirchhoff relations has the following standard form:

$$H = \begin{pmatrix} Y & 0 & 1_{n_p} & A^T \\ -A & 1_{n_q} & 0 & 0 \end{pmatrix} \sigma_S$$

with the following additional constraints:

$$Y\vec{1} = 0, \quad A\vec{1} = \vec{1}.$$

Here $\vec{1}$ is a column vector of all 1's.

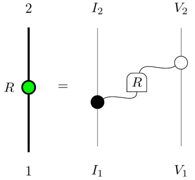
$A\vec{1} = \vec{1}$ says A is **quasi-stochastic** as the rows of A sum to 1 but can contain negatives.

Subcategories I

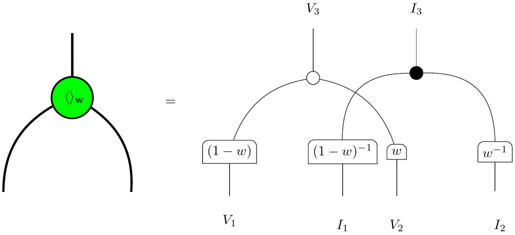
- Lagrangian relations satisfying the Kirchhoff current law form a subcategory, $\mathsf{KirRel}_F \subseteq \mathsf{LagRel}_F$.
- Kirchhoff relations include resistor circuits and they also allow additional new components: namely **ideal current dividers** ($A\vec{1} = \vec{1}$).

Theorem

All maps in Kirchhoff relations KirRel_F *are generated by the current divider, resistors and junctions.*



Resistor



Current Divider

Deterministic Kirchhoff Relations

Lemma

A deterministic Kirchhoff relation over F, has a parity-check matrix in standard Kirchhoff form, with the additional constraint that A is deterministic. (A is deterministic if there is only 1 one in each row)

- This corresponds to the subprop ResRel (resistor circuits) which was studied by Baez and Fong [4].

[4] Baez, John C., and Brendan Fong. "A compositional framework for passive linear networks." arXiv preprint arXiv:1504.05625 (2015)

Power

- Power associated to a state $\mathcal{R}$ is given as follows:

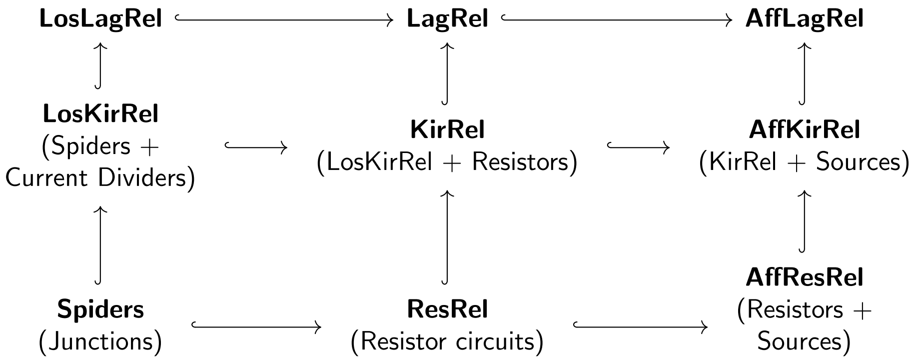
$$P = -q^T Y q$$

- $\mathcal{R}$ is said to be **lossless** if $P_{\mathcal{R}} = 0$.

Lemma

1. *For $\mathcal{R}$ to be lossless $q^T Y q$ must vanish, which implies that $Y = 0$.*
2. *Any state in* LossKirRel *has $Y = 0$ and $A\vec{1}_{n_p} = \vec{1}_{n_q}$.*

Interrelation of Categories



Quantum Connection: $\textbf{AffLagRel}_F$ = **Qudit Stabilizer Quantum Mechanics**

The Road ahead...

- Connections to qudit error correction...
- Importing techniques of electrical network theory to quantum circuits.
- Normal forms for electrical circuits?
- Normal Form for stabilizer circuits?

References

- Comfort, Cole, and Aleks Kissinger. "A Graphical Calculus for Lagrangian Relations." arXiv preprint arXiv:2105.06244 (2021).
- Baez, John C., and Brendan Fong. "A compositional framework for passive linear networks." arXiv preprint arXiv:1504.05625 (2015).
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- Pawel Sobocinski blog on Graphical Linear Algebra: https://graphicallinearalgebra.net/