Dynamical Systems via Domains

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Institute for Logic, Language and Computation

Introduction I

- Domain theory provided a mathematical semantics for 'symbolic' computation
- Lack this explainability for 'non-symbolic' computation (e.g., neural nets)
- Today: extend domain-theoretic semantics to non-symbolic computation

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- Non-symbolic computation as a dynamical system:
  - (computational) states
  - dynamics (program)
- Task: build 'domain' describing its behavior.

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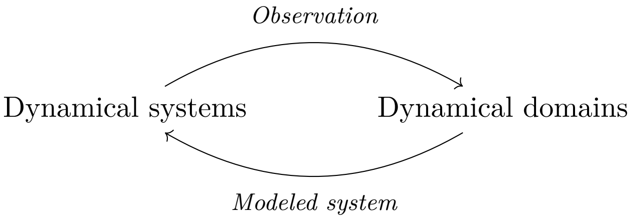

Outline

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From systems to domains

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Definition (a very general one)

A dynamical system $\mathfrak{X}$ is a structure $(X, \mathcal{A}, \mu, T)$ where

- state space $(X, \mathcal{A}, \mu)$ is probability space (standard Borel)
- dynamics $T : X \to X$ is measurable.

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Example: machine learning

- At each training stage, machine characterized by a set w of parameters/weights.
- Given some data d, the learning algorithm produces $w' = L(w, d)$.
- As a dynamical system:
  - State space $X = W \times D^{\omega}$
  - Dynamics $T : X \to X$ maps

$$\Big(w, \delta\Big) \mapsto \Big(L(w, \delta_0)\ ,\ (\delta_1, \delta_2, \ldots)\Big)$$

- A measure on X describing random initialization and data sampling needn't be preserved by T.

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System morphisms

Definition

A system morphism (or factor)

$$\varphi : (X, \mathcal{A}, \mu, T) \to (Y, \mathcal{B}, \nu, S)$$

is a partial function $\varphi : X \to Y$ with domain $M \subseteq X$ and codomain $N \subseteq Y$ such that

- M and N are invariant sets of full measure.
- Measure-preserving: $\mu(\varphi^{-1}(B)) = \nu(B)$
- Equivariant: $\varphi(T(x)) = S(\varphi(x))$.

If φ bijective, it is an isomorphism. Two morphisms identified if identical on invariant set of full measure.

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- and the order describes information containment.
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- A directed-complete partial order (dcpo) is a partial order $(D, \leq)$ where every directed subset A has a least upper bound $\bigvee A$ (aka join).
- Scott topology: $U \subseteq D$ is Scott-open if
  - $a \in U, a \leq b \Rightarrow b \in U$
  - $A \subseteq D$ directed, $\bigvee A \in U \Rightarrow \exists a \in A : a \in U$
- Function $f : D \to E$ between dcpos Scott-continuous iff monotone and preserves directed joins.
- Scott domain: non-empty, 'ω-algebraic', 'bounded complete' dcpo.

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- Intuitively, $\mathfrak{D}$ consists of 'basic' elements that represent increasingly finer observations of $\mathfrak{X}$,
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Observed system I

- $A \in \mathcal{A}$ as observation/measurement: if system in a state $x \in A$, measurement A is positive.
- A finite cover $\mathcal{C} \subseteq \mathcal{A}$ of X, yields observed system reflecting the original one:

- For observation parameter $i = (n, \mathcal{C})$ and $x \in X$ define

$$\mathcal{O}_i(x) = \left\{ t \in \mathcal{C}^n : T^k(x) \in t_k \text{ for } k = 0, \ldots, n-1 \right\}$$

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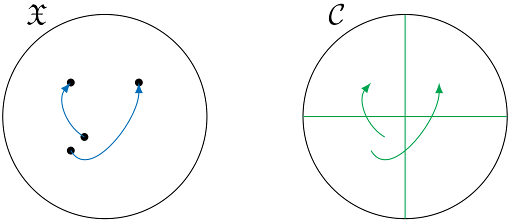

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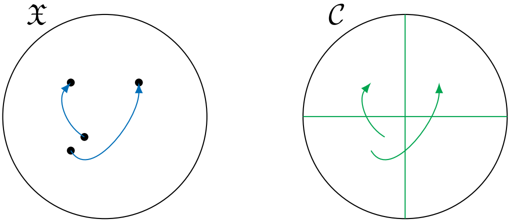

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Observed system II

- Set of observable behaviors $\mathsf{H}_i := \{\mathcal{O}_i(x) : x \in X\}$ has natural dynamics:

$$\mathcal{O}_i(x) \mapsto \mathcal{O}_i(T(x)).$$

- But not functional $\rightsquigarrow$ Smyth powerdomain:
  - $D_i := \{M : \emptyset \neq M \subseteq \mathsf{H}_i\}$, ordered by $\supseteq$
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  - $f_i : D_i \to D_i$ by $f_i(M) := \{\mathcal{O}_i(T(y)) : \mathcal{O}_i(y) \in M\}$
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  - $v_i(\mathcal{O}_i(x)) := \mu\{y \in X : \mathcal{O}_i(y) = \mathcal{O}_i(x)\}$.
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- Idea: Refine observation parameters i and take the limit of the $\mathfrak{D}_i$.
- Won't use all of $\mathcal{A}$, only measurements from a countable subset $\mathcal{B}$ 'generating' $\mathcal{A}$.
- Say $\mathcal{B} \subseteq \mathcal{A}$ is a basis if closed under finite intersection.
- Write $I(\mathcal{B})$ for the set of observation parameters $i = (n, \mathcal{C})$ with $\mathcal{C} \subseteq \mathcal{B}$.

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- Then $(I(\mathcal{B}), \leq)$ is a directed preorder
- for $i \leq j$ have $p_{ij} : D_j \to D_i$ with

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1. Fix background category C of domains (dcpos with least element, Scott-continuous functions).
2. Subscript f: full subcategory of finite domains.
3. Superscript p: wide subcategory where morphisms also are projections.
4. Define expanding system as diagram in $\mathsf{C}_\mathsf{f}^\mathsf{p}$ with additional properties (directed index set).
5. Show expanding systems have limits in C^p.
6. Define desired category D (bifinite domains) as full subcategory of C whose objects are limits of expanding systems.

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Definition

A dynamical dcpo $\mathfrak{D}$ is a triple (D, v, f) where

- D is a dcpo
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$\mathfrak{D}$ is a dynamical Scott domain if, additionally,

- D is a Scott domain
- v max-normalized ($v(D) = 1$ and $\max D$ is countable intersection of Scot-opens with v-value 1).

Background category II

Definition

A dynamical morphism $\alpha : (D, v, f) \to (E, w, g)$ is a Scott-continuous function $\alpha : D \to E$ such that

1. Max-preserving: $\alpha(\max D) \subseteq \max E$.
2. Max-bisimulative: If $a \in D$ and $e \in \max E$ with $\alpha(a) \leq e$, there is $a \leq d \in \max D$ with $\alpha(d) = e$.
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- Background category dSCO: dynamical Scott domains with dynamical morphisms.

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Expanding system

Definition

An expanding system is a diagram $(\mathfrak{D}_i, p_{ij})_{I^{\mathrm{op}}}$ in $\mathsf{dSCO}_{\mathsf{f}}^{\mathsf{p}}$ where I is a countable directed preorder and

- upward deterministic: if f_i fails, on input a_i, to uniquely pick out a maximal element above it, this will be eventually remedied.

Formally: For all $i \in I$, if $\exists a_i, b_i \neq b'_i \in \max D_i : b_i, b'_i \geq f_i(a_i)$, then there is $j \geq i$ in I such that $\forall a_j, b_j, b'_j \in \max D_j$:

- *if $p_{ij}(a_j) = a_i, p_{ij}(b_j) = b_i, p_{ij}(b'_j) = b'_i$,*
- *then $b_j \not\geq f_j(a_j)$ or $b'_j \not\geq f_j(a_j)$.*

The limit theorem I

Theorem

Let $(\mathfrak{D}_i, p_{ij})_I$ *be an expanding system. Then*

1. $D := \{a \in \prod_{i \in I} D_i : a(i) = p_{ij}(a(j))\}$ *is Scott domain. Define* $p_i : D \to D_i$ *by* $p_i(a) := a(i)$.
2. *There is a unique max-normalizing valuation* v *with* $v_i(U) = v(p_i^{-1}(U))$.
3. *There is a pointwise largest* $f : D \to D$ *that is Scott-continuous, max-preserving,* $f(a)(i) \geq f_i(a(i))$.

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3. *There is a pointwise largest* $f : D \to D$ *that is Scott-continuous, max-preserving,* $f(a)(i) \geq f_i(a(i))$.

And $(\mathfrak{D}, p_i)$ *with* $\mathfrak{D} := (D, v, f)$ *is the 'restricted' limit:*

- *a cone in* dSCO^p *with max-preserving domain*
- *any other cone with max-preserving domain uniquely factors through it.*

The limit theorem II

- Restricted limit:
  - Want $\mathfrak{D}$ to model a dynamical system on its maximal elements, so f should be max-preserving.
  - $\mathfrak{D}$ limit subject to being max-preserving.
- Example:
  - The diagram $(\mathfrak{D}_i, p_{ij})$ from observing a system.
- Proof:
  - The construction of D is standard [AJ94].
  - For v one can use existing work [GL18].
  - The difficulty is with f, bit of a tour de force.

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Category of dynamical domains

Definition

- A dynamical domain is a dynamical Scott domain $\mathfrak{D}$ that is the restricted limit of an expanding system.
- The full subcategory of dSCO whose objects are dynamical domains is denoted dDOM.
- So morphisms in dDOM are dynamical morphisms (not required to be projections).

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Outline

From domains to systems

Theorem

Let $\mathfrak{D} = (D, v, f)$ *be a dynamical domain. Then*

1. $\max D$ *with the relative Scott topology* τ *is a compact 0-dim Polish space*
2. f *restricts to a continuous function on* $\max D$
3. v *determines a probability measure* μ_v *on* $\mathcal{B}(\tau)$.

Thus, we obtain the dynamical system

$$\mathsf{S}(\mathfrak{D}) := \big(\max D, \mathcal{B}(\tau), \mu_v, f \restriction \max D\big).$$

Outline

Putting everything together

Theorem

- *Let* $\mathfrak{X} = (X, \mathcal{A}, \mu, T)$ *be a dynamical system.*
- *Let* $\mathcal{B} \subseteq \mathcal{A}$ *be countable basis that is separating.*
- *Build* $\mathfrak{D}_i = (D_i, v_i, f_i)$ *and* $p_{ij} : D_j \to D_i$ *as described.*
- *Then* $(\mathfrak{D}_i, p_{ij})_{I(\mathcal{B})}$ *is an expanding system.*
- *Let observation domain* $\mathsf{D}(\mathfrak{X}, \mathcal{B}) := (D, v, f)$ *be the dynamical domain obtained as restricted limit.*
- *The canonical embedding of* $\mathfrak{X}$ *into* $\mathsf{S}(\mathsf{D}(\mathfrak{X}, \mathcal{B}))$

$$\varphi : X \to \max D \qquad x \mapsto \left\langle \{\mathcal{O}_i(x)\} : i \in I(\mathcal{B}) \right\rangle$$

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- Idea: The operations D and S should be adjoint.
- Use category TS_{0c} of topological systems:
  - Objects: (X, τ, μ, T) with X 0-dim compact Polish space, μ probability measure on $\mathcal{B}(\tau)$, T continuous.
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- Restricts to equivalence: demand for $\mathfrak{D}$
  - For $a \in D$, $a = \bigwedge(\uparrow a \cap \max D)$
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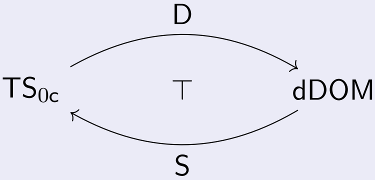

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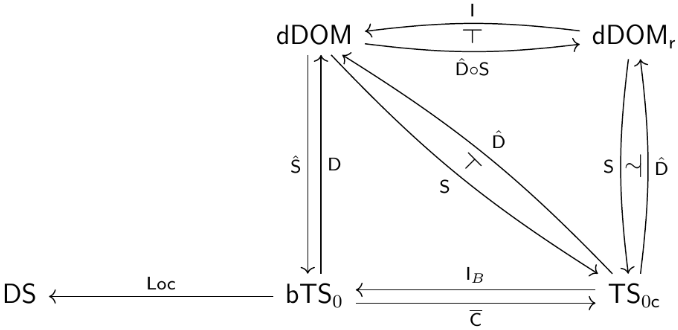
Detailed picture



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- Reap the benefits: study dynamical systems through the rich domain theory (e.g., entropy).
- Relation symbolic vs. non-symbolic computation:
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- Computability theory for non-symbolic computation via effective domain theory.
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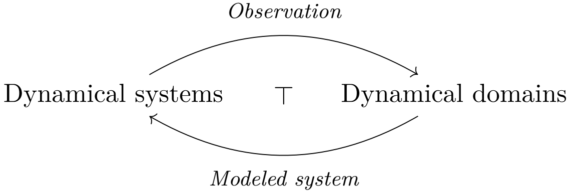

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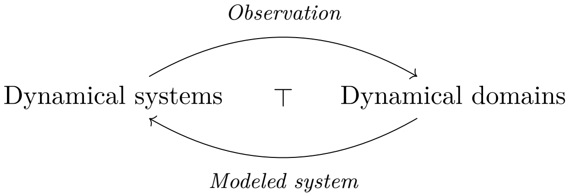

Thank you!

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