On the pre- and promonoidal structure of spacetime

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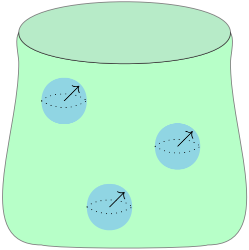
University of Oxford

ACT 2022

Compositionality in CQM



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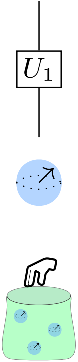

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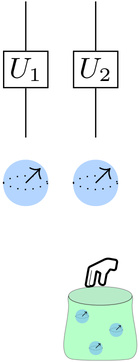

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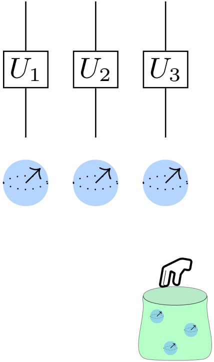

Decompositionality

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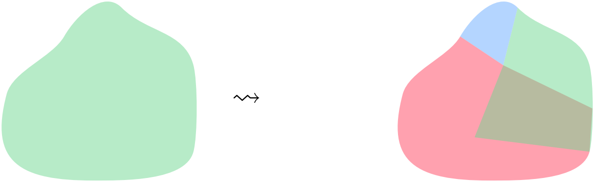

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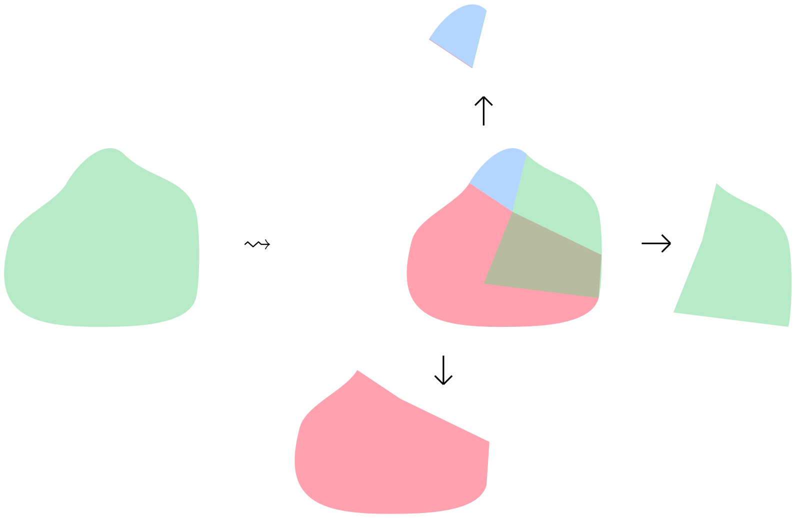
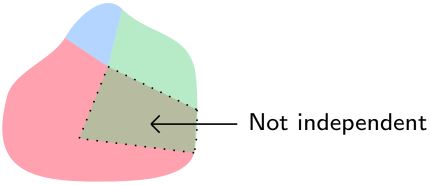


Two problems:

1. 

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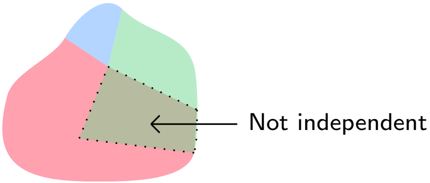
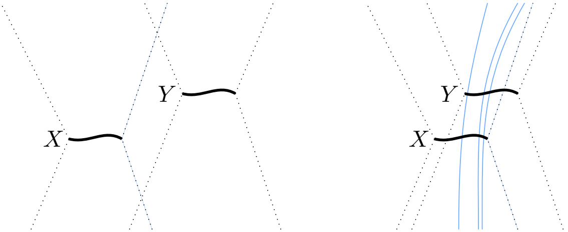
1. 

2. Joint system might not be well-defined



$X \otimes Y : \checkmark$ $\qquad$ $X \otimes Y : ?$

Monoidal Categories

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
$$I \in \mathsf{Ob}(\mathcal{C})$$

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$$I \otimes A \overset{\lambda}{\cong} A \overset{\rho}{\cong} A \otimes I$$

[Submitted on 29 Jul 2011]

Causal categories: relativistically interacting processes

Bob Coecke, Raymond Lal

[Submitted on 30 May 2019]

A Process-Theoretic Church of the Larger Hilbert Space

Stefano Gogioso

[Submitted on 30 Mar 2020]

Functorial evolution of quantum fields

Stefano Gogioso, Maria E. Stasinou, Bob Coecke

Partially Monoidal Categories

Definition (Partial Functor)

$$F : \mathcal{C} \rightharpoonup \mathcal{D}$$

where:

$$\mathcal{C} \xhookleftarrow{F_0} \mathcal{S} \xrightarrow{F_1} \mathcal{D}$$

$\mathcal{S}$ full, replete subcategory of $\mathcal{C}$.

Coecke & Lal, 2013, Found Phys 43. Gogioso, 2019, arXiv:1905.13117

Partially Monoidal Categories

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Can we “totalise” the tensor?

Idea:

1. Premonoidal categories to deal with non-independence of systems

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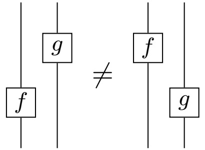
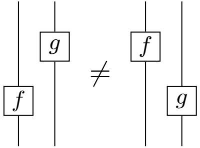
2. Promonoidal categories to deal with ill-defined joint systems

Premonoidal Categories



Power & Robinson, 1997, Math Struct in CS 7(5)

Premonoidal Categories



$$\otimes : \mathcal{C} \,\square\, \mathcal{C} \to \mathcal{C}$$

$$I \in \mathsf{Ob}(\mathcal{C})$$

Central isomorphisms:

$$(A \otimes B) \otimes C \overset{\alpha}{\cong} A \otimes (B \otimes C)$$

$$I \otimes A \overset{\lambda}{\cong} A \overset{\rho}{\cong} A \otimes I$$

Power & Robinson, 1997, Math Struct in CS 7(5)

Promonoidal Categories

Definition (Profunctor)

$P : \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $P : \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$, equivalently $\hat{P} : \mathcal{C} \to [\mathcal{D}^{\mathsf{op}}, \mathsf{Set}]$.

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$$(Q \circ P)(-, =) = \int^{d} Q(-, d) \times P(d, =)$$

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Units:

$$F(-) \cong \int^{c} Fc \times \mathcal{C}(c, -), \qquad G(-) \cong \int^{c} \mathcal{C}(-, c) \times Gc$$

Promonoidal Categories

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Day, 1970, In: Reports of the Midwest Category Seminar IV

Promonoidal Categories

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$$\otimes(1 \times \otimes) \overset{\alpha}{\cong} \otimes(\otimes \times 1)$$

$$\otimes(J \times 1) \overset{\lambda}{\cong} 1 \overset{\rho}{\cong} \otimes(1 \times J)$$

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Promonoidal Categories

Joint systems are given by presheaves

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When representable

$$(A \otimes B)(-) \cong \mathcal{C}(-, X)$$

we can identify "$A \otimes B \cong X$"

A Toy Category of Slices

Fix $\mathcal{M}$, a connected time-orientable Lorentzian manifold

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Definition (Slice)

A closed set $S \subset \mathcal{M}$ where no $x \neq y$ in S are connected by a future-directed causal curve.

A Toy Category of Slices

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Definition (Slice)

A closed set $S \subset \mathcal{M}$ where no $x \neq y$ in S are connected by a future-directed causal curve.

Definition (Jointly Spacelike Slices)

X and Y jointly spacelike if $X \cup Y$ is a slice.

A Toy Category of Slices

Definition

Slice category with:

Objects	Slices of $\mathcal{M}$
Morphisms	$\gamma : X \to Y$ set of causal curves through X then Y
Composition	$\delta \circ \gamma = \delta \cap \gamma$
Identities	$1_X : X \to X$ set of all causal curves through X

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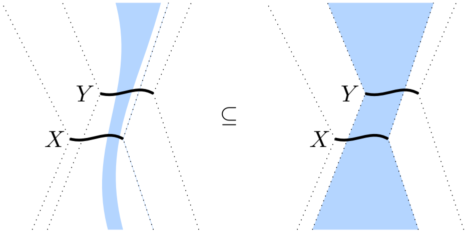
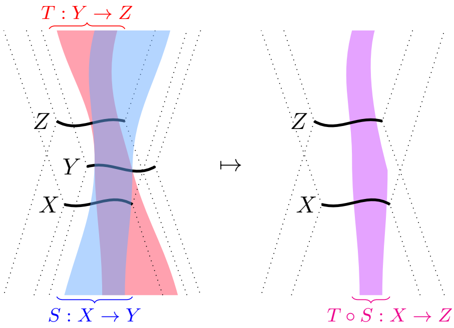


$$f : X \to Y$$

A Toy Category of Slices



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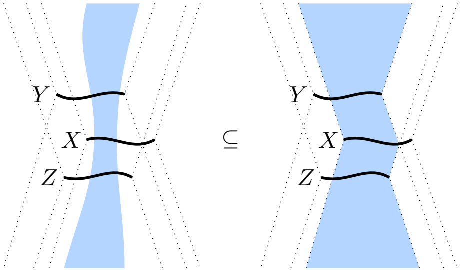

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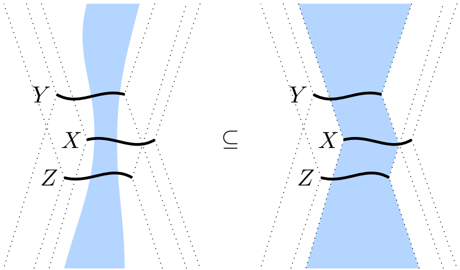

$\owedge$ extends to a profunctor $\owedge : \mathsf{Slice} \times \mathsf{Slice} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathsf{Slice}$

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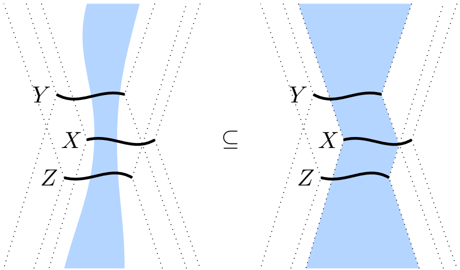

$\oslash$ extends to a profunctor $\oslash$: Slice × Slice ⇸ Slice

$$J(-) := \mathcal{P}(\mathcal{C}[-]) : \mathsf{Slice}^{\mathsf{op}} \to \mathsf{Set}$$

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Theorem
$(\mathsf{Slice}, \owedge, J)$ *is a symmetric promonoidal category.*

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Theorem
$(X \owedge Y)(-)$ *is representable if and only if* X *and* Y *are jointly spacelike.*

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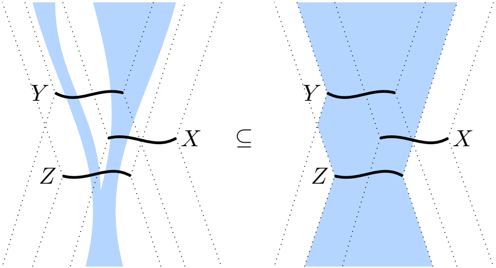

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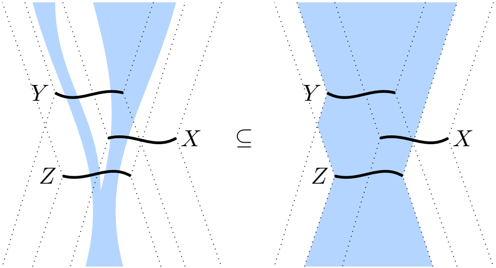

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Conjecture

$(\mathsf{Slice}, \varovee, \mathsf{Slice}(-, \varnothing))$ *is a symmetric "pro-premonoidal" category.*

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Theorem

For every slice X*,* $(X \ovee -)(-)$ *is a multiplicative kernel for* $(\mathsf{Slice}, \owedge, J)$*.*

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- How does the story play out with enrichment?