A category-theoretic proof of the Ergodic Decomposition Theorem

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Joint work with Sean Moss

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Applied Category Theory 2022

Ergodicity

Idea:

An *ergodic system* is a situation where "all the mass is well mixed".

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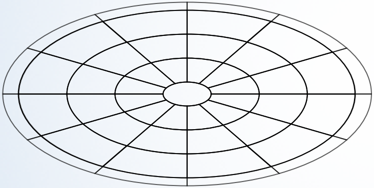

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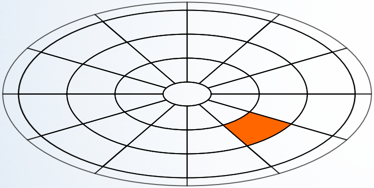

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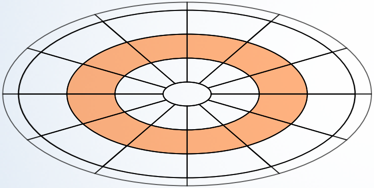

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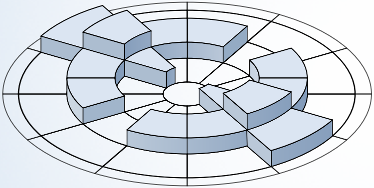

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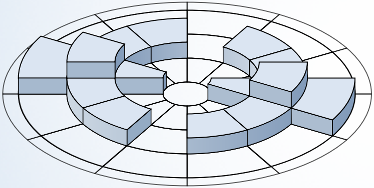

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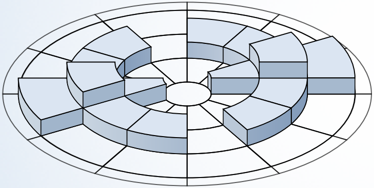

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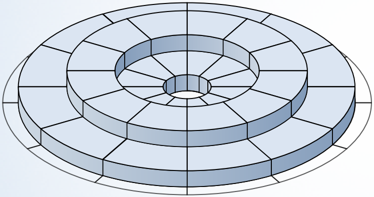

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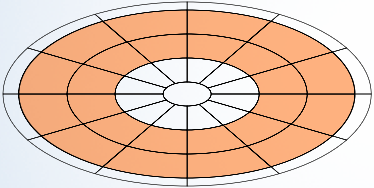

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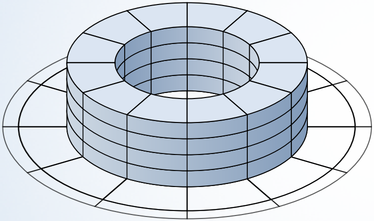

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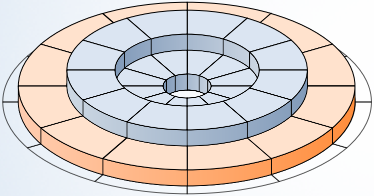

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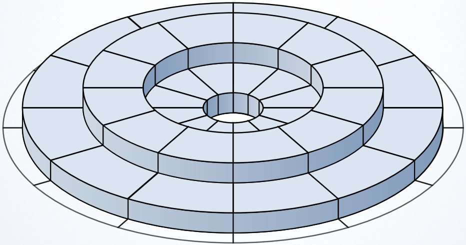
Ergodic Decomposition Theorem:

Every invariant measure is a mixture of ergodic ones.



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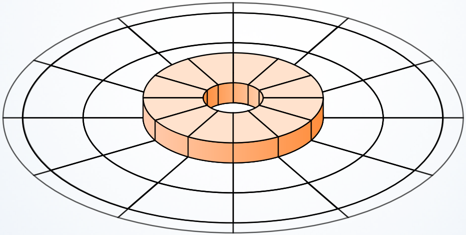

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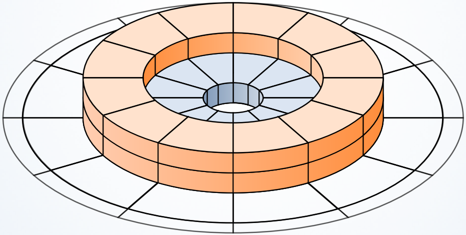

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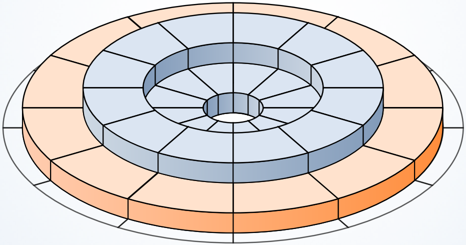

Markov categories

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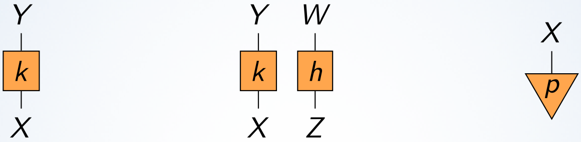

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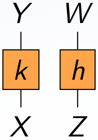

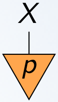

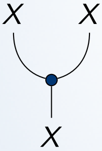

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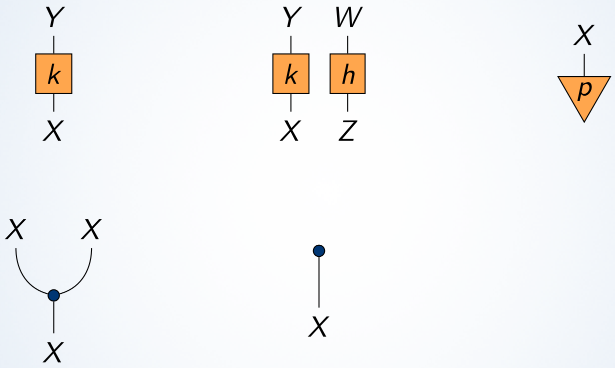

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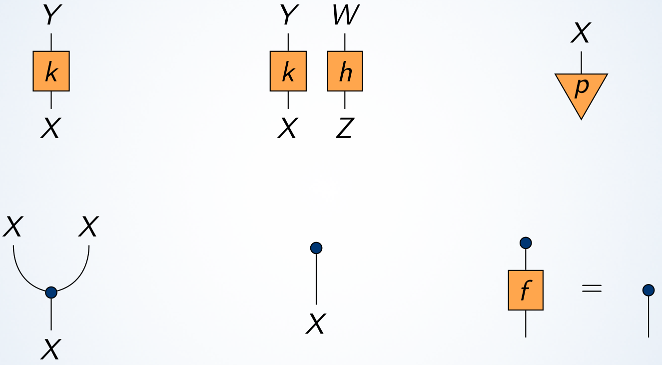


Markov categories

The category FinStoch

- Objects are finite sets (or natural numbers);
- Morphisms are stochastic matrices $p : X \to Y$ of entries $p(y|x)$;

$$q \circ p(z|x) = \sum_y q(z|y)\, p(y|x)$$

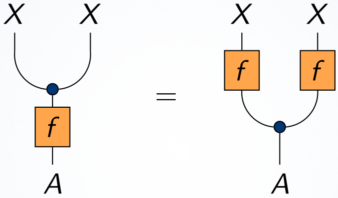
The category Stoch

- Objects are measurable spaces;
- Morphisms are Markov kernels $k : X \to Y$ of entries $p(B|x)$;

$$h \circ k(C|x) = \int_Y h(C|y)\, k(dy|x)$$

Markov categories

Deterministic morphisms:



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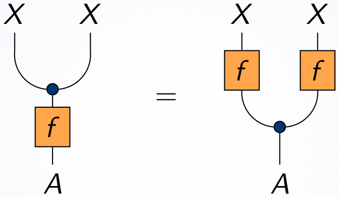

- In FinStoch, these are the matrices with only entries 0 and 1.

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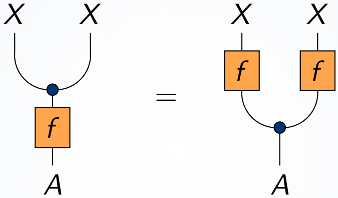
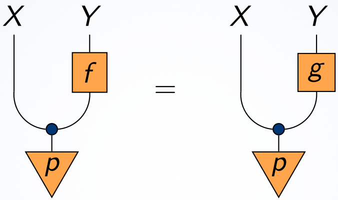


- In FinStoch, these are the matrices with only entries 0 and 1.
- In Stoch, these are the kernels indexing measures with only values 0 and 1 for each measurable set.

Markov categories

Almost-sure equality:



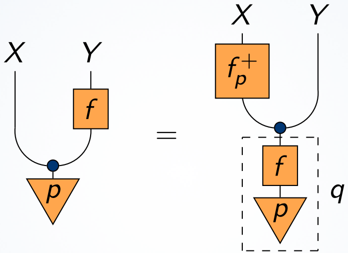
- In FinStoch, this means that $f = g$ on the support of p.
- In Stoch,

$$\int_A f(B|x)\, p(dx) = \int_A g(B|x)\, p(dx)$$

i.e. f and g differ only on a set of p-measure zero.

Markov categories

Conditionals, Bayesian inverses, disintegrations:



- In FinStoch: $p(x)\, f(y|x) = q(y)\, f_p^+(x|y)$
- In Stoch:

$$\int_A f(B|x)\, p(dx) = \int_B f_p^+(A|y)\, q(dy).$$

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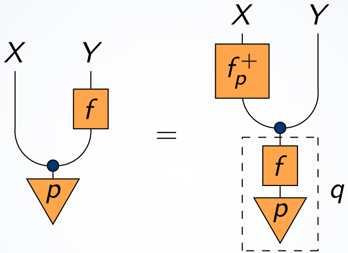

- If f_p^+ exists for all p and deterministic f, we say that X satisfies a *disintegration theorem*. (E.g. if X is standard Borel.) In Stoch, conditional expectations assemble to a *regular conditional*.

Markov categories

Decomposition of states:

$$p(x) = \lambda_1 \, q_1(x) + \lambda_2 \, q_2(x) \qquad \lambda_i \geq 0, \quad \lambda_1 + \lambda_2 = 1$$

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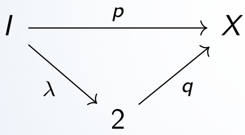
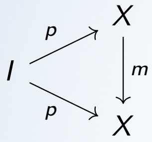


$$p(x) = \sum_{i \in 2} q(x|i)\, \lambda(i)$$

Dynamical systems with Markov categories

Invariant states:



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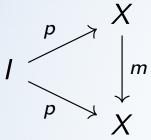

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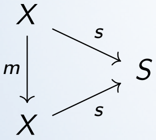
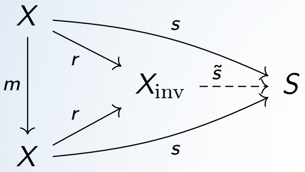
Invariant (deterministic) observables:



In Stoch these correspond to invariant sets (especially for $S = 2$).

Dynamical systems with Markov categories

Markov colimits:



In Stoch, if m acts deterministically, X_{inv} is given by the *sigma-algebra of invariant sets*.

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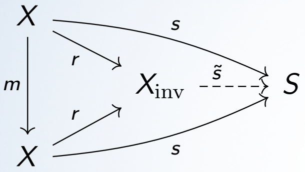

In Stoch, if m acts deterministically, X_{inv} is given by the *sigma-algebra of invariant sets*.

Ergodic states:

An invariant stat $I \xrightarrow{p} X$ is *ergodic* if and only if for each invariant observable s,

$$I \xrightarrow{\;p\;} X \xrightarrow{\;s\;} Y$$

is deterministic. In Stoch: zero-one on invariant sets!

Main statement

Theorem:
Let C be a Markov category. Let X be a deterministic dynamical system in C with monoid M. Suppose that

- The underlying object X of C has disintegrations;
- The Markov colimit X_{inv} of the dynamical system exists.

Then every invariant state of X can be written as a composition $k \circ q$ such that k is q-almost surely ergodic.

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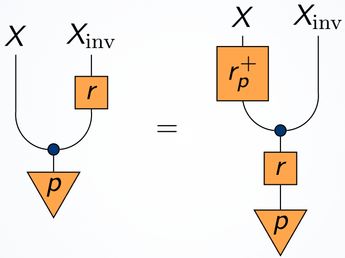
Corollary:

Let X be a measurable dynamical system on a space that admits disintegrations. Then every invariant measure on X can be written as a mixture of ergodic ones.

Main statement

Proof:



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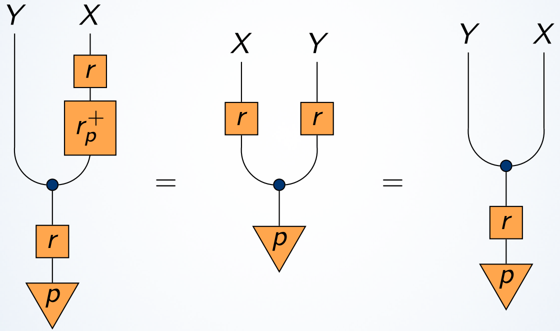

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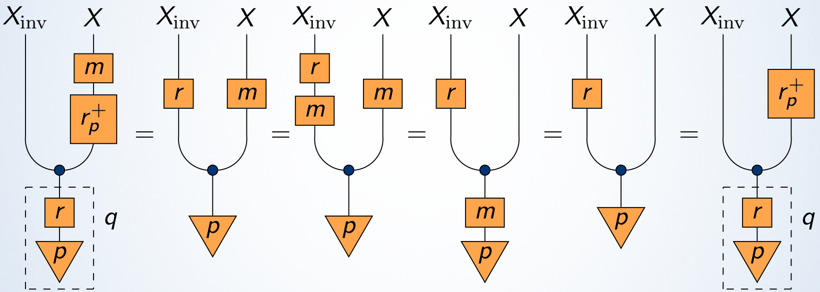

Conclusion

- This is the first step into ergodic theory with Markov categories
- The definition of ergodic measure is very natural in this framework
- Convex mixtures of measures are just categorical composition
- The ergodic decomposition theorem has a conceptual proof in terms of string diagrams. Measure theory is “outsourced” to e.g. the disintegration requirement.
- Coming next: ergodic theorems, stochastic dynamical systems!

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