Universality of asymmetric lens proxy pullbacks

THE UNIVERSITY *of* EDINBURGH

Applied Category Theory 2022

Matthew Di Meglio

With	Index	Kind	Message
Ava	0	sent	Hi
	1	received	Hey
	2	sent	What's new?
	3	draft	I'm g
Cam	0	received	Hey

Current Time	12:49

Time	From	To	Message
09:00	Matt	Ava	Hi
11:11	Bob	Ava	Gday
11:15	Ava	Matt	Hey
12:23	Matt	Ava	What's new?
12:31	Cam	Matt	Hello

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Asymmetric delta lens

Asymmetric delta lens

Source

$$\begin{tikzcd} S \arrow[d, "F"'] \\ V \end{tikzcd}$$

View

Asymmetric delta lens

Source

$$\begin{tikzcd} S \arrow[d, "F"'] \\ V \end{tikzcd}$$

View

Get functor

Asymmetric delta lens

X

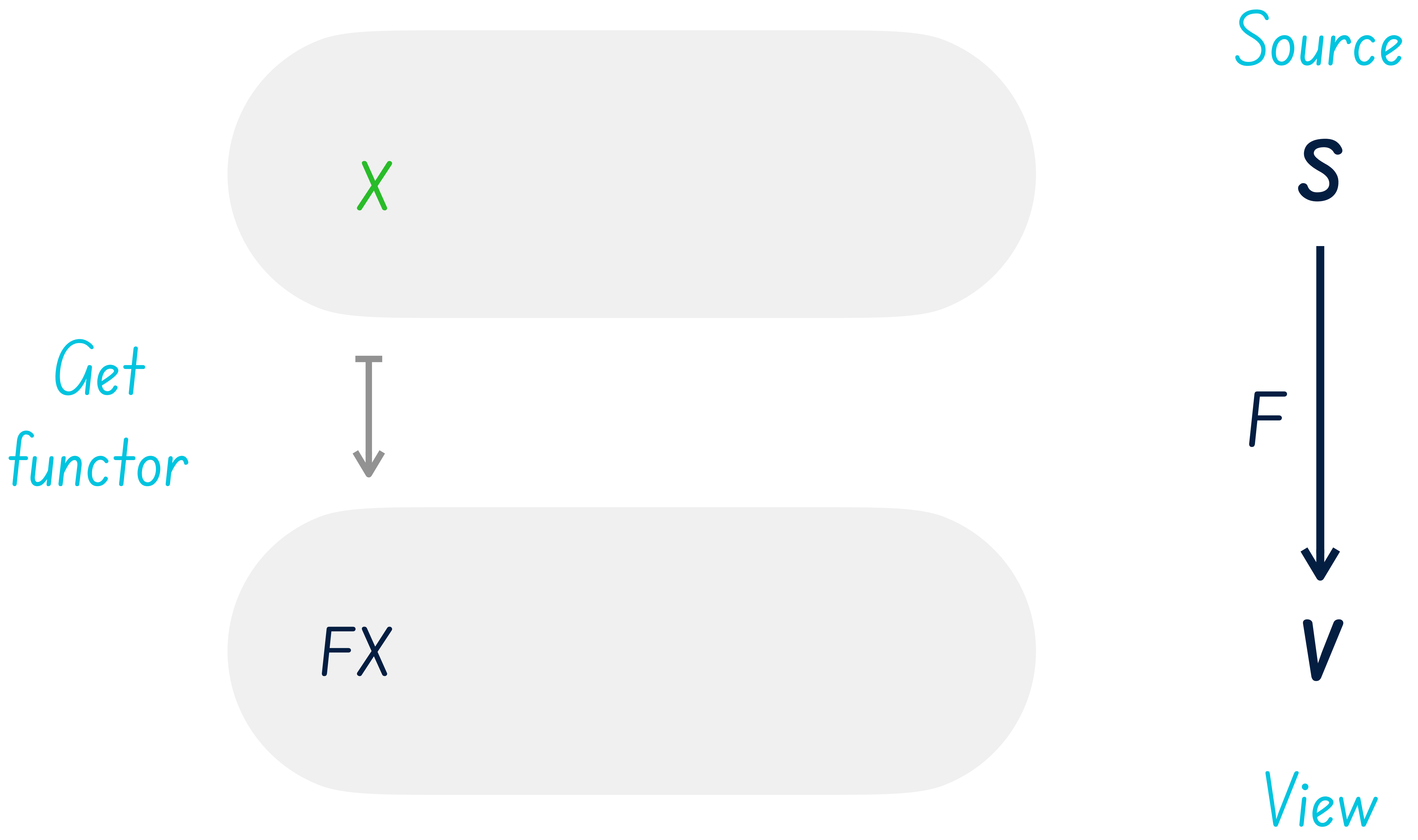
Source

$$\begin{tikzcd} S \arrow[d, "F"'] \\ V \end{tikzcd}$$

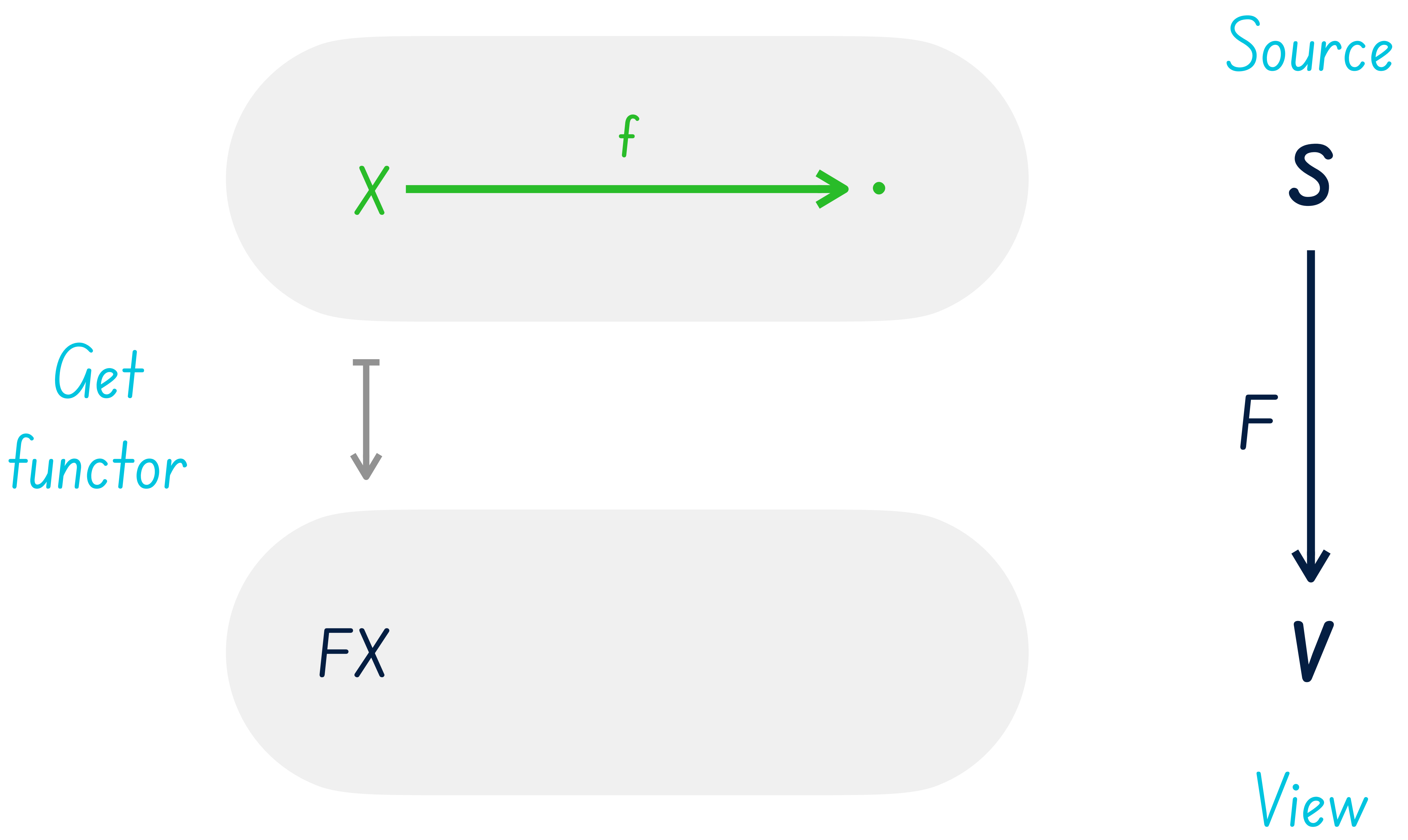
View

Get
functor

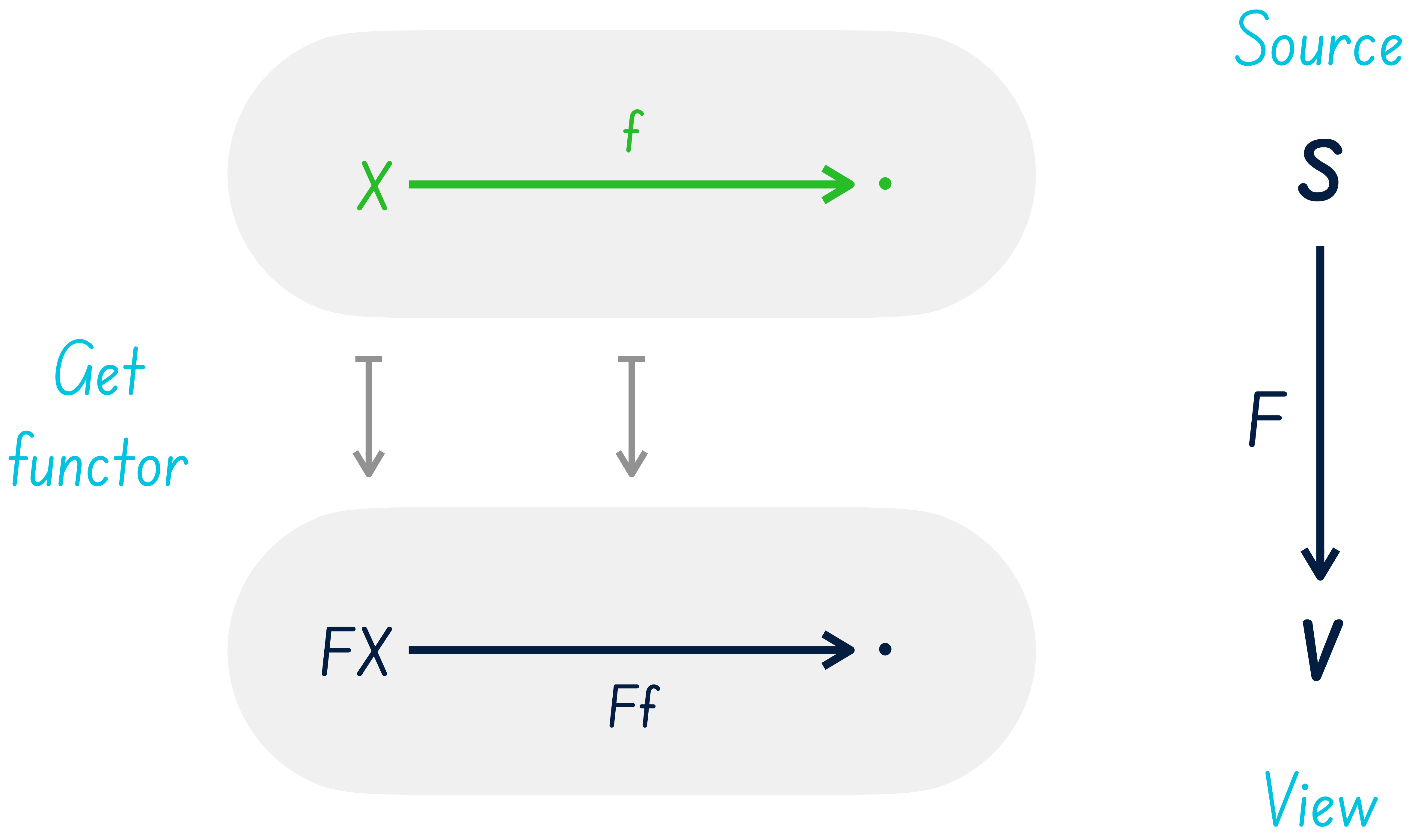
Asymmetric delta lens



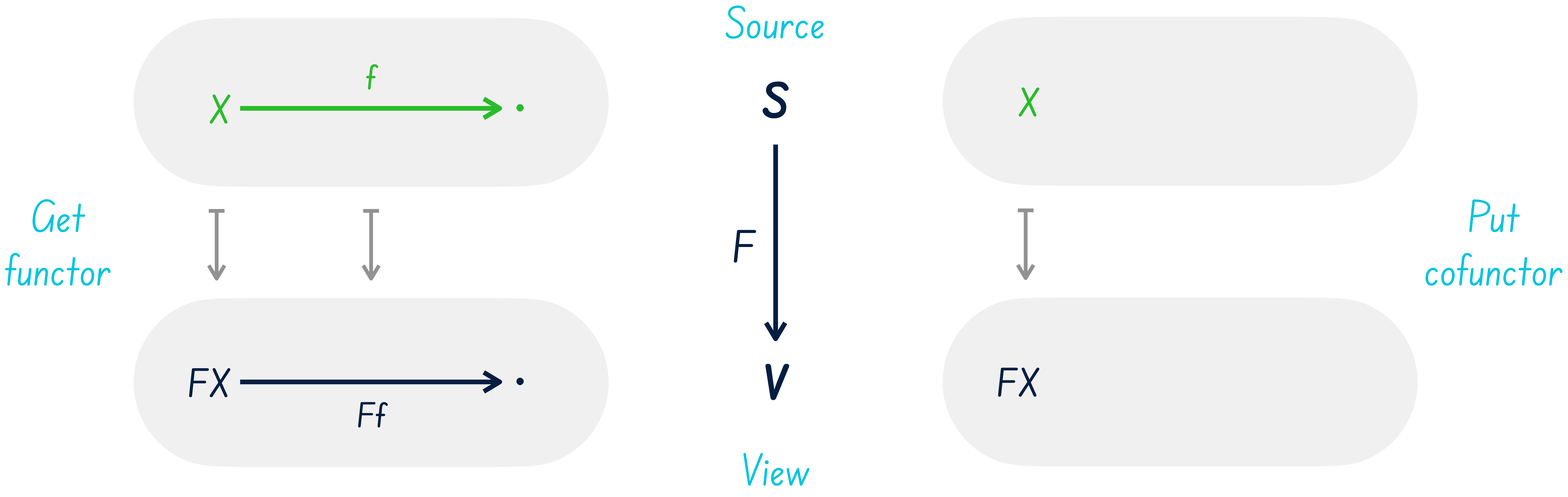
Asymmetric delta lens



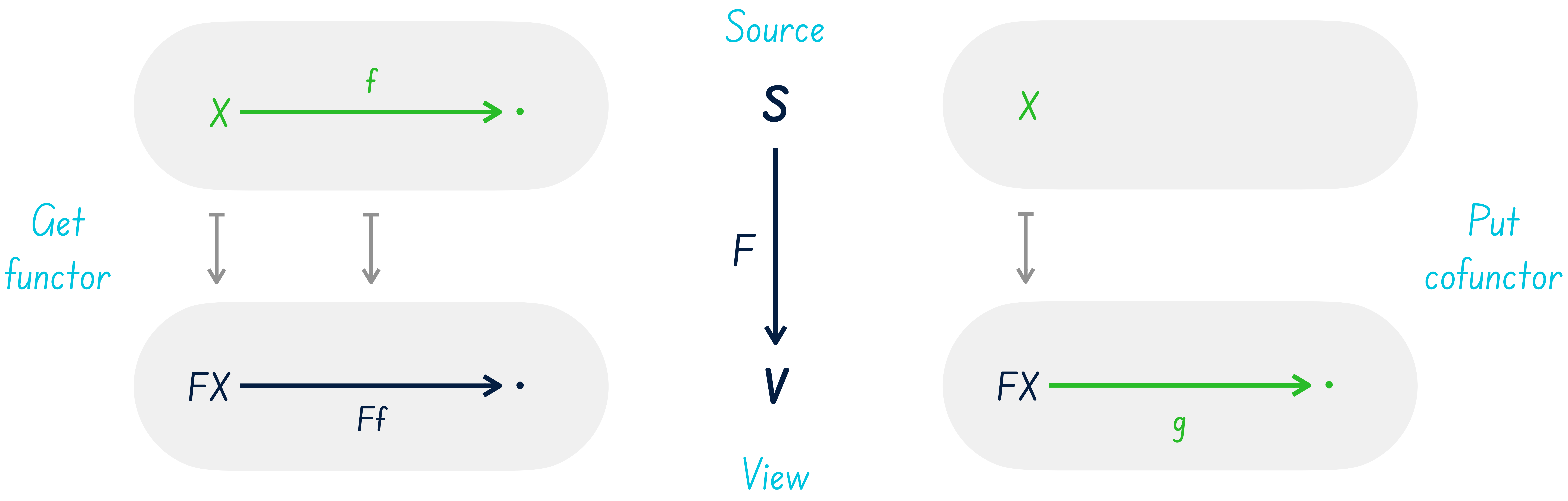
Asymmetric delta lens



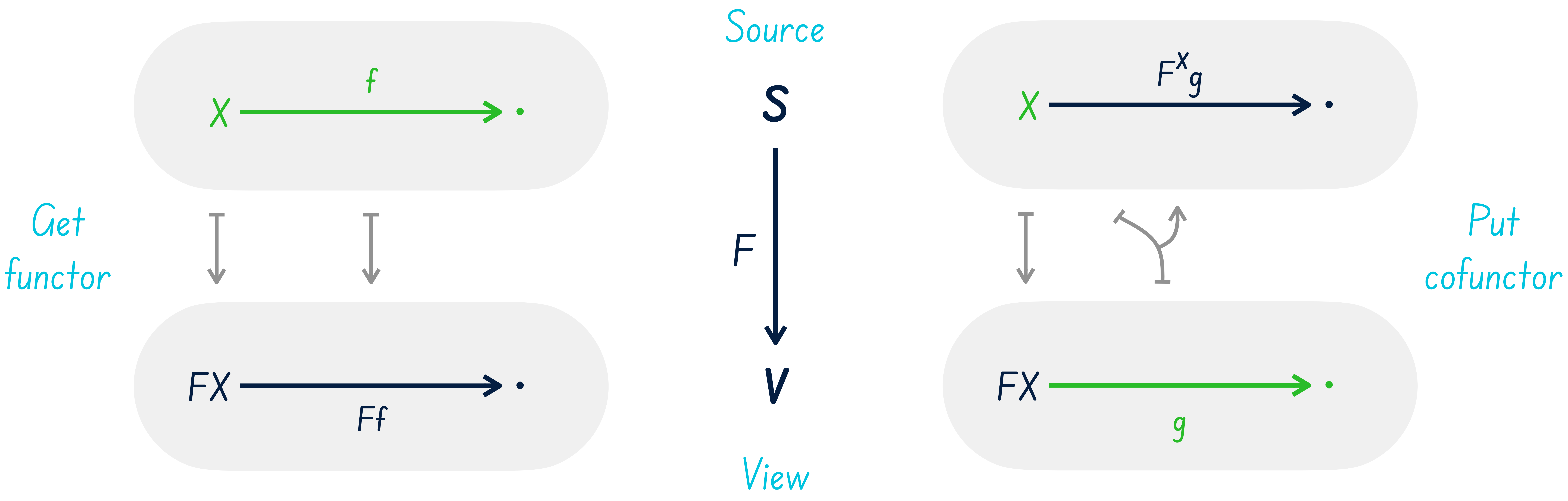
Asymmetric delta lens



Asymmetric delta lens



Asymmetric delta lens



Lens spans model bidirectional transformations

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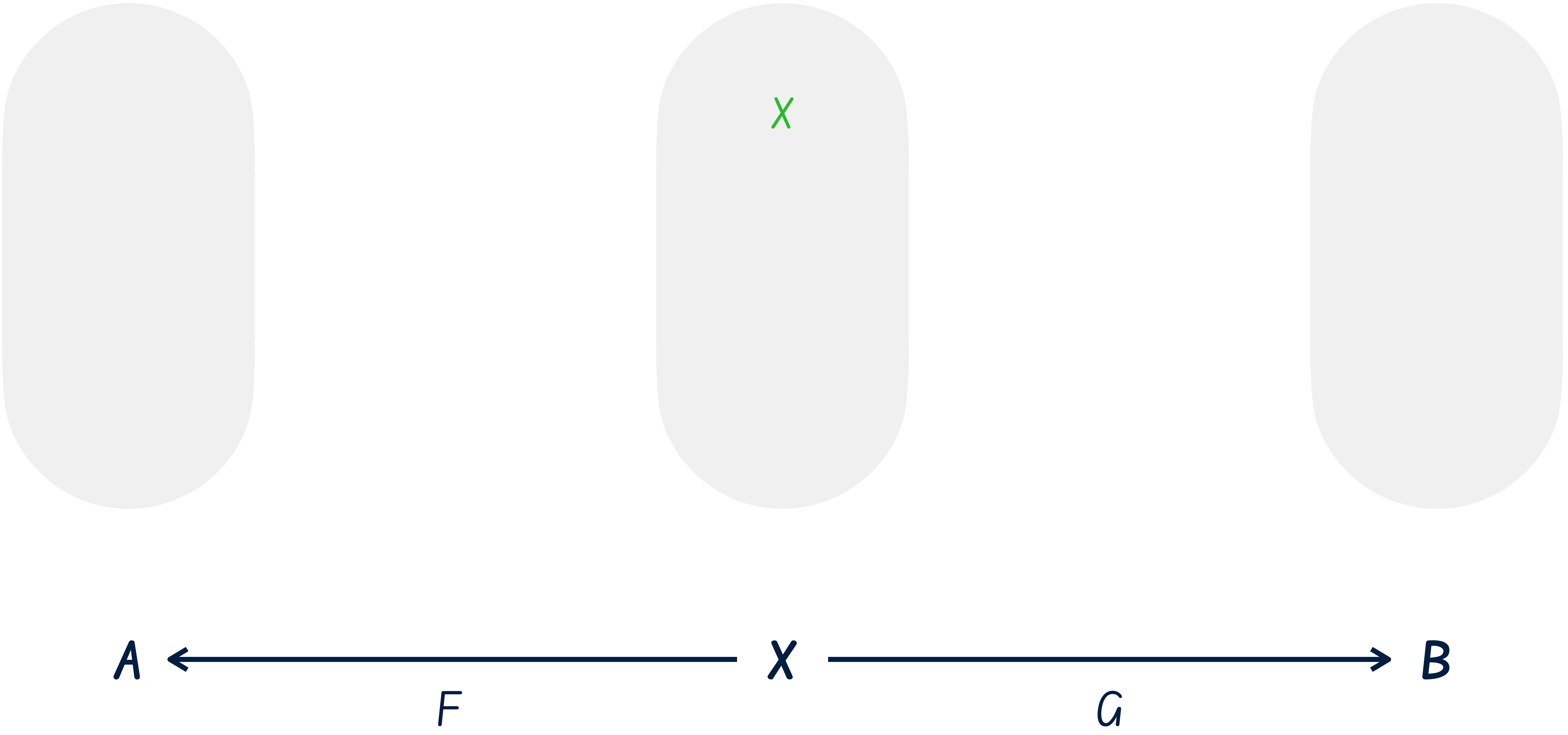
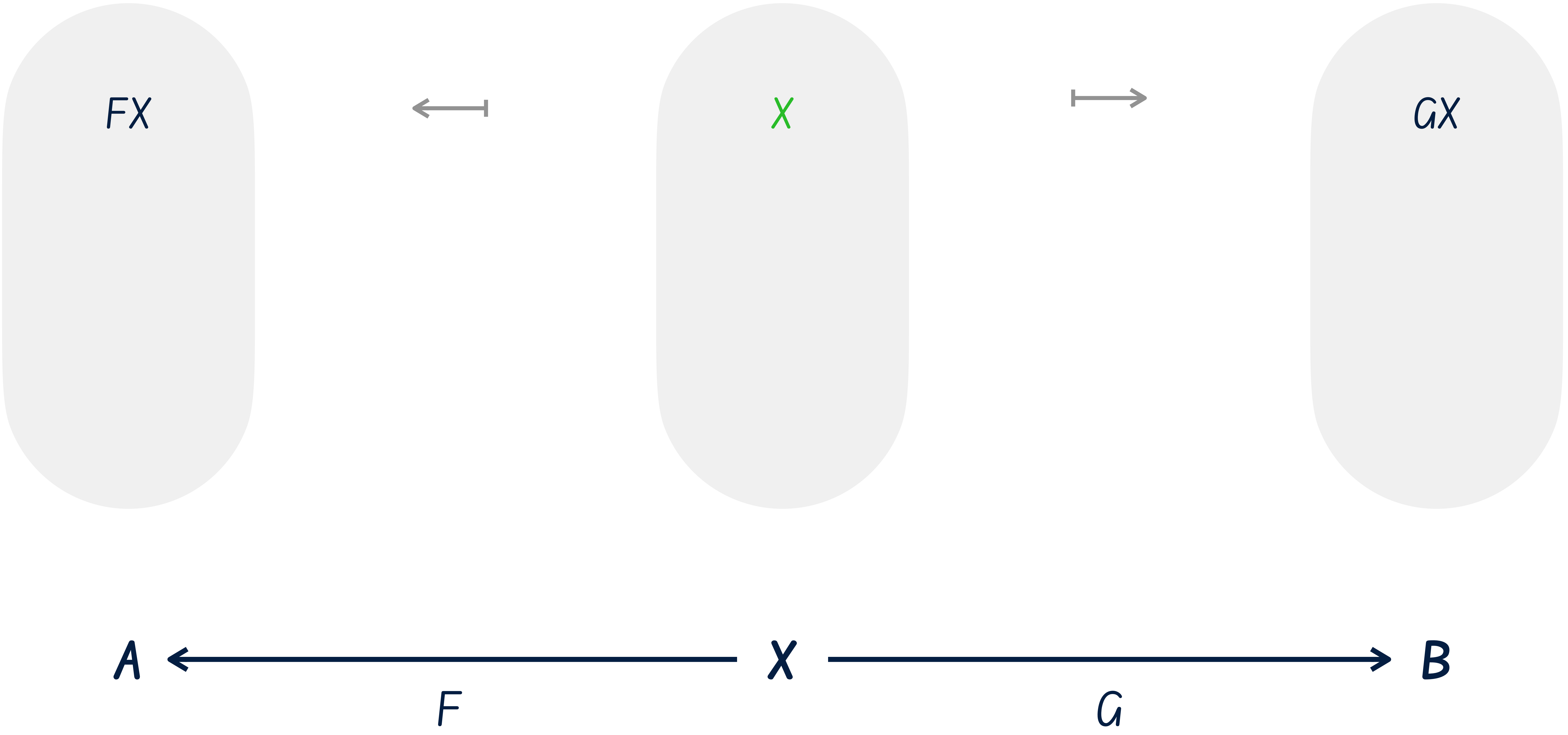
$$A \xleftarrow{F} X \xrightarrow{G} B$$

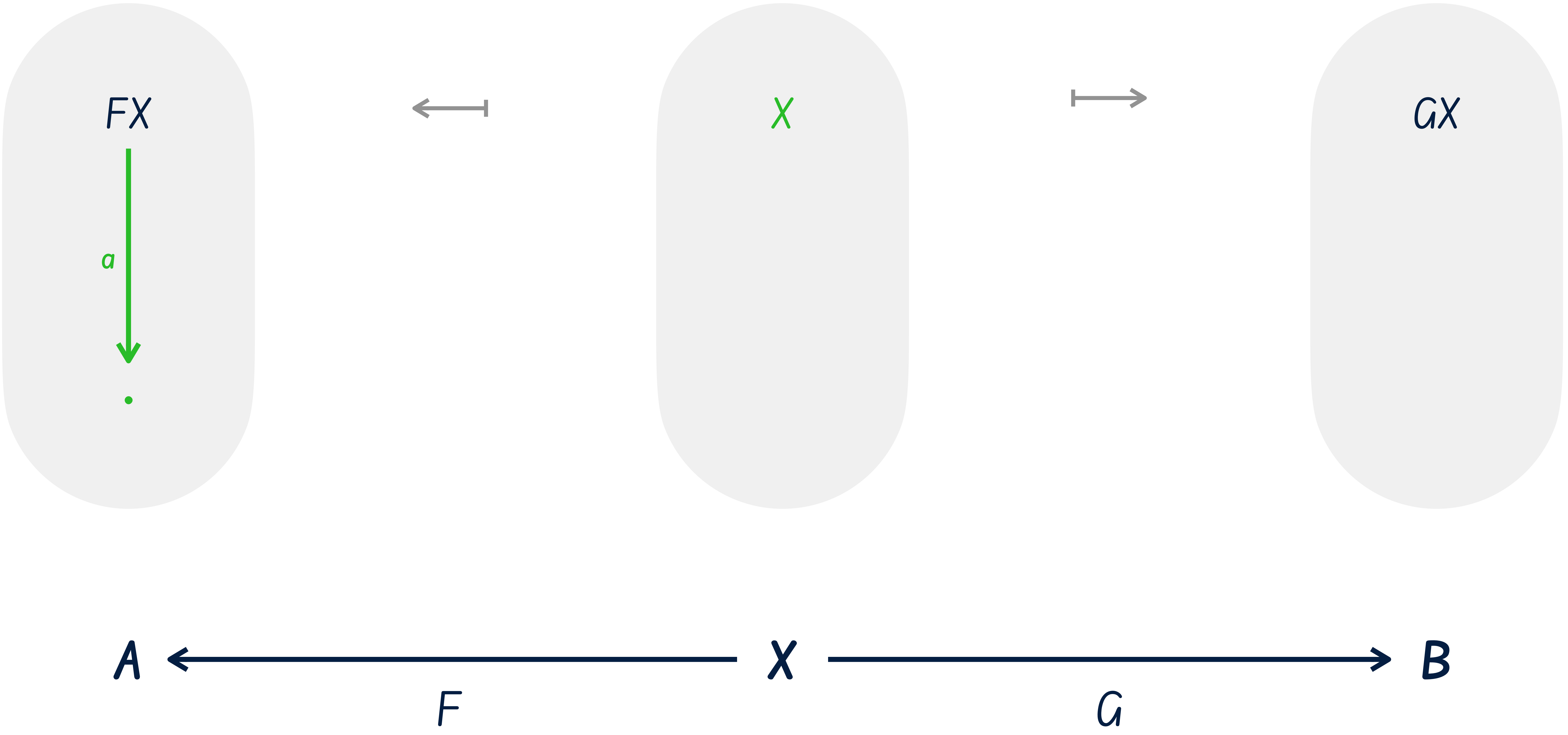
Lens spans model bidirectional transformations



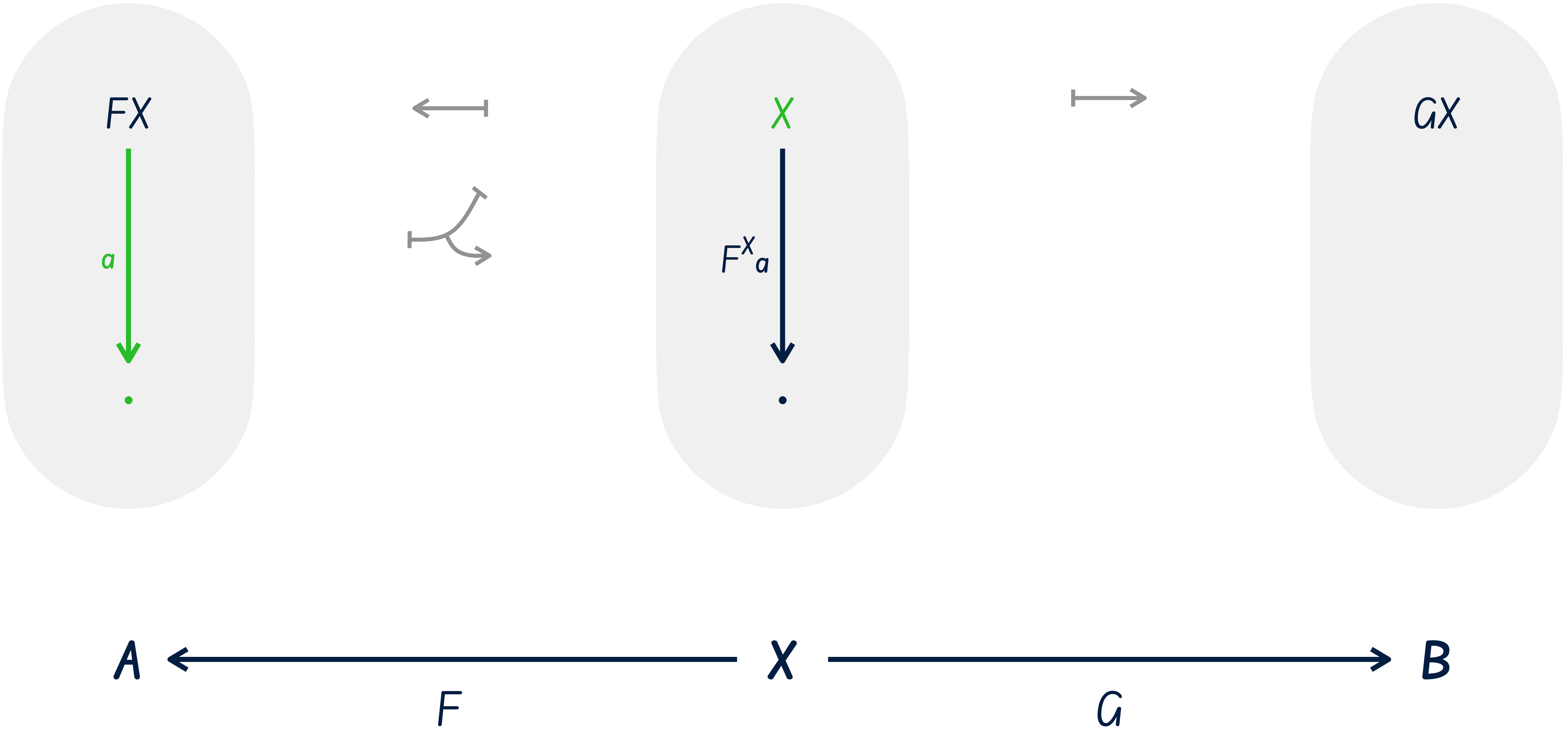
Lens spans model bidirectional transformations



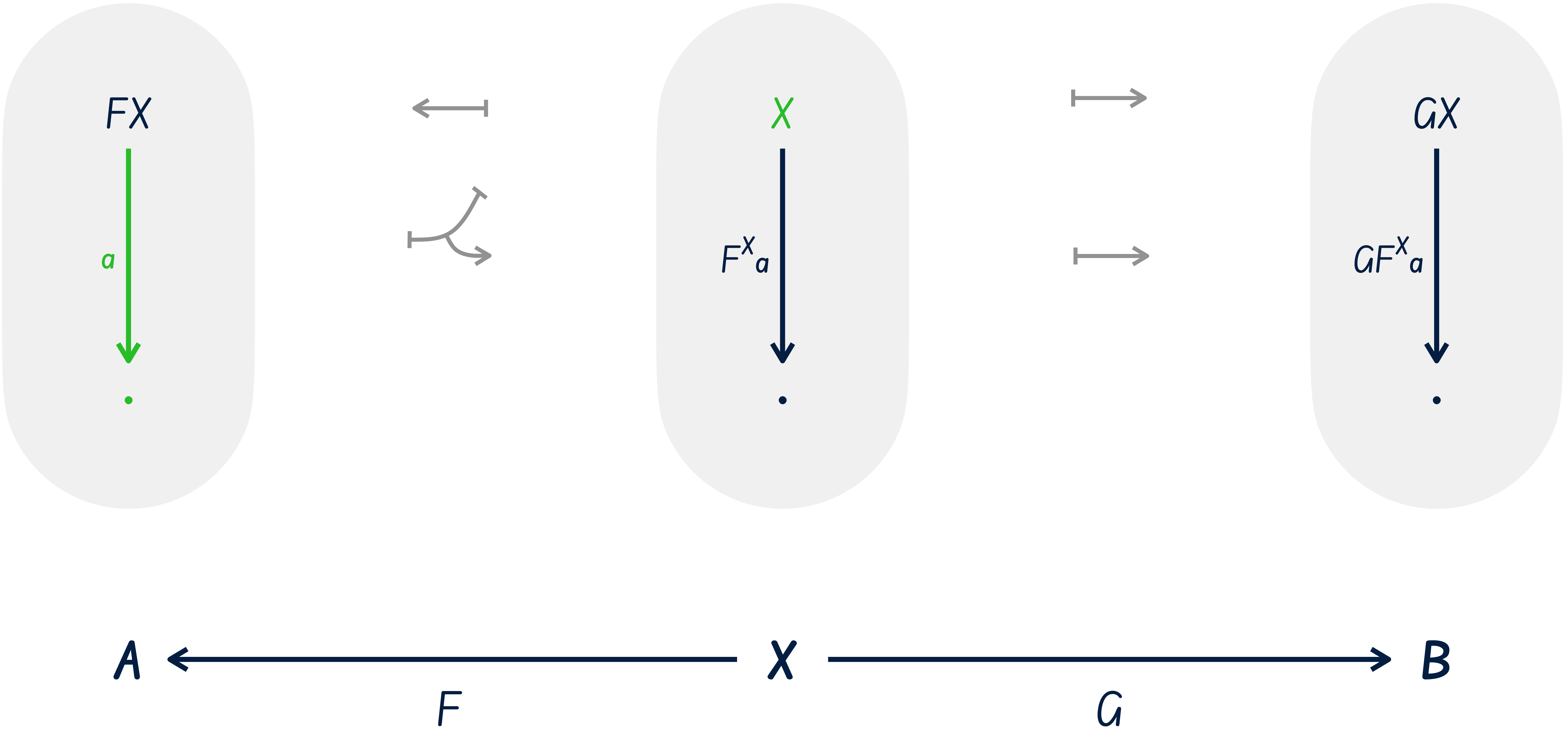
Lens spans model bidirectional transformations



Lens spans model bidirectional transformations



Lens spans model bidirectional transformations



Span composition by pullback?

Span composition by pullback?

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\ D & & C & & E \end{tikzcd}$$

Span composition by pullback?

$$
\begin{tikzcd}
& & P \arrow[dl, dashed] \arrow[dr, dashed] & & \\
& A \arrow[dl] \arrow[dr] & & B \arrow[dl] \arrow[dr] & \\
D & & C & & E
\end{tikzcd}
$$

Span composition by pullback?

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

Span composition by pullback?

$$
\begin{tikzcd}
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
& C &
\end{tikzcd}
$$

Span composition by pullback?

$$\begin{tikzcd} & P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\ A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd}$$

Span composition by pullback?

(A,B)

A

$$\begin{tikzcd}
 & P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}$$

Span composition by pullback?

(A,B)

$A \xrightarrow{a} \cdot$

$$
\begin{tikzcd}
& P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
& C &
\end{tikzcd}
$$

Span composition by pullback?

$$(A,B) \xrightarrow{(?,\ ?)} \cdot$$

$$A \xrightarrow{a} \cdot$$

$$\begin{tikzcd}
& P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
& C &
\end{tikzcd}$$

Span composition by pullback?

$$(A,B) \xrightarrow{(a,\ ?)} \cdot$$

$$A \xrightarrow{a} \cdot$$

$$\begin{tikzcd}
 & P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
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\end{tikzcd}$$

Span composition by pullback?

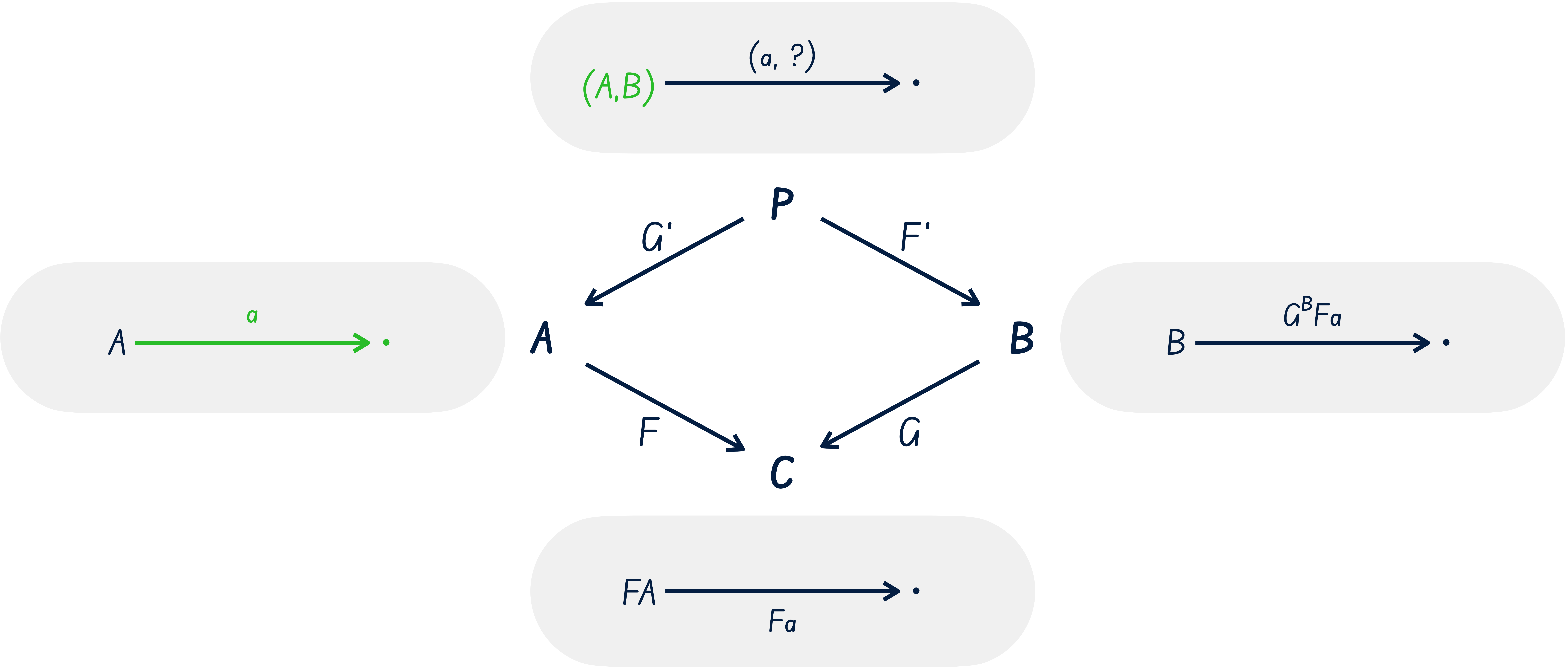
$$(A,B) \xrightarrow{(a,\ ?)} \cdot$$

$$A \xrightarrow{a} \cdot$$

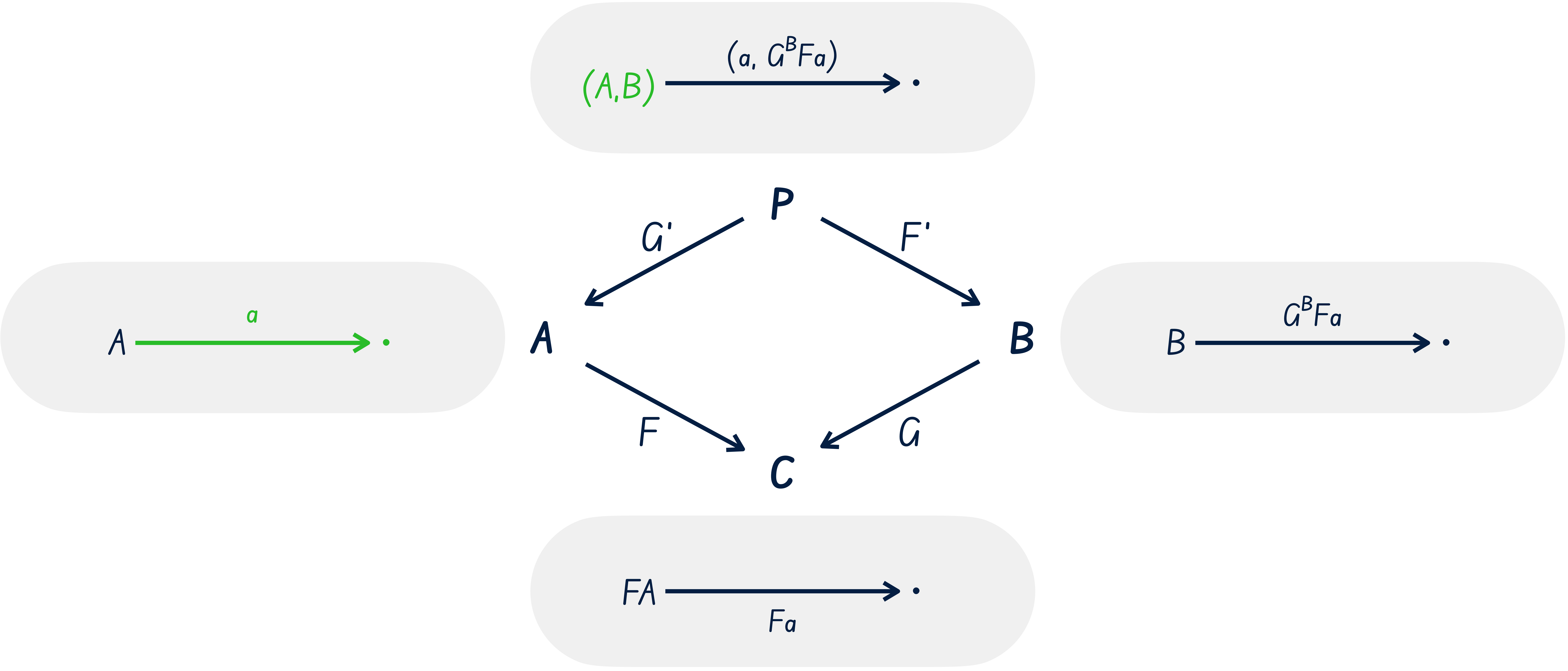
$$\begin{tikzcd} & P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\ A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd}$$

$$FA \xrightarrow{Fa} \cdot$$

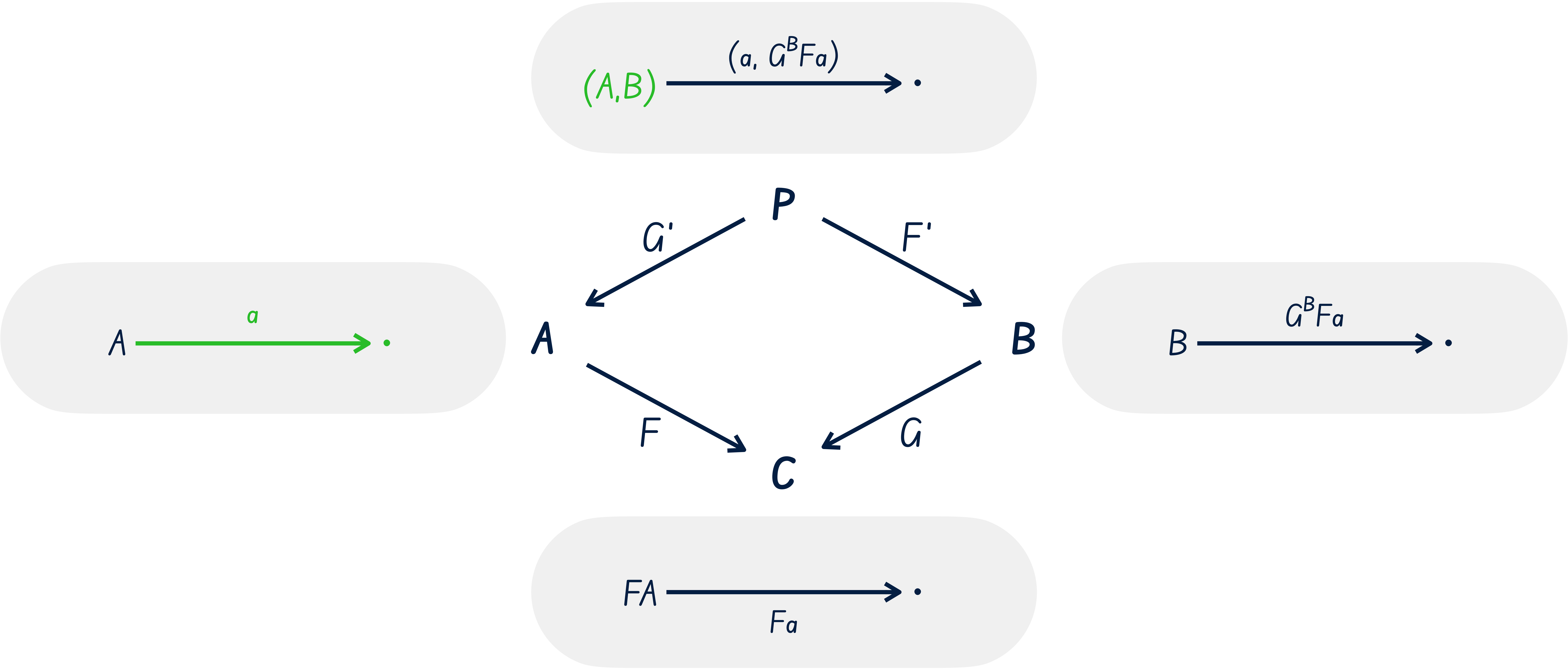
Span composition by pullback?



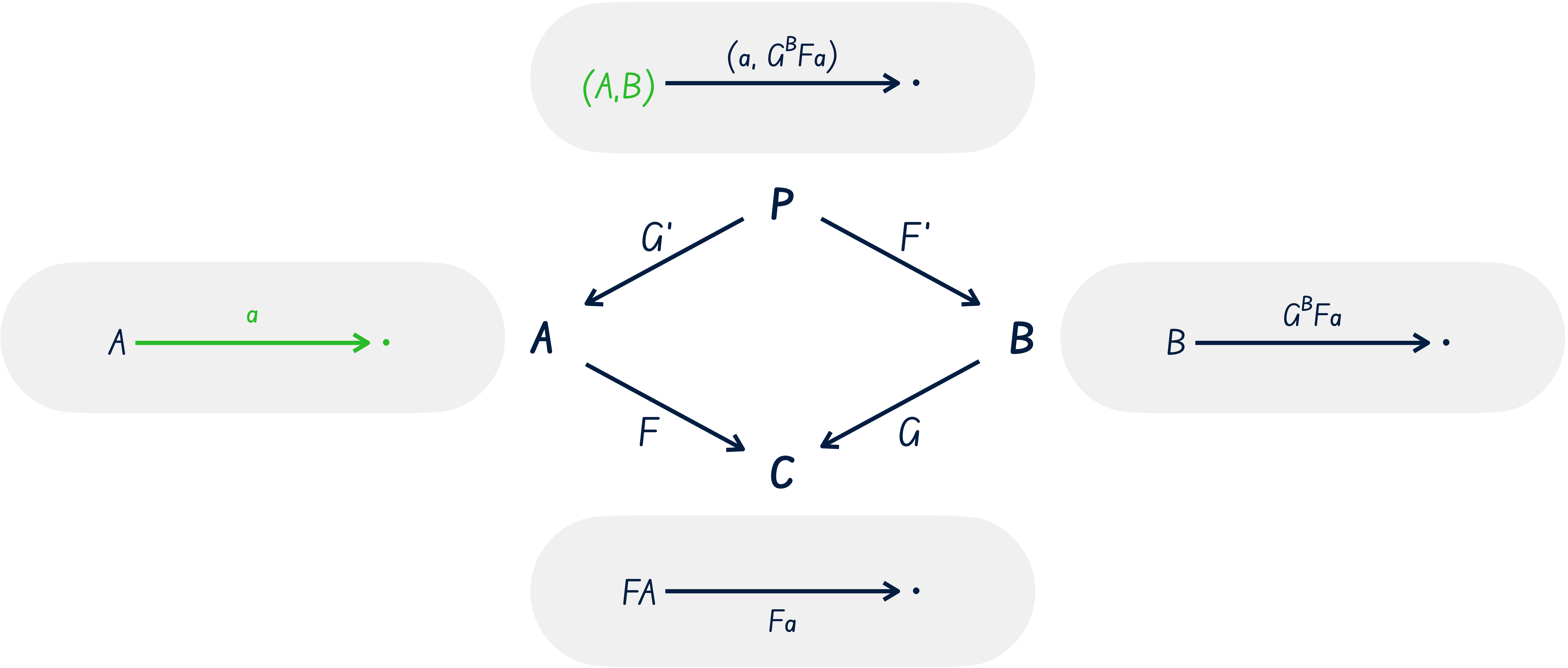
Span composition by pullback?



Span composition by ~~pullback~~?



Span composition by ~~pullback~~ proxy pullback?



$$
\begin{tikzcd}
& P \arrow[dl, "G'"'] \arrow[dr, "F'"] \arrow[dd, phantom, "\text{Proxy Pullback}"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
& C &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & D \arrow[ddl, bend right, "K"'] \arrow[ddr, bend left, "J"] & \\
 & P \arrow[dl, "G'"'] \arrow[dr, "F'"] \arrow[dd, phantom, "\text{Proxy Pullback}"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & D \arrow[dl, bend right, "K"'] \arrow[d, dashed, "L"] \arrow[dr, bend left, "J"] & \\
 & P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
A \arrow[dr, "F"'] & \text{Proxy Pullback} & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}
$$

Necessary conditions

If L exists then

$$
\begin{tikzcd}
 & D \arrow[d, dashed, "L"] \arrow[ddl, bend right, "K"'] \arrow[ddr, bend left, "J"] & \\
 & P \arrow[dl, "G'"'] \arrow[dr, "F'"] \arrow[dd, phantom, "\text{Proxy Pullback}"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}
$$

Necessary conditions

If L exists then

1. (K,J) is **compatible** with (F,G)

$$
\begin{tikzcd}
 & D \arrow[ddl, bend right, "K"'] \arrow[d, dashed, "L"] \arrow[ddr, bend left, "J"] & \\
 & P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
A \arrow[dr, "F"'] & \text{Proxy Pullback} & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}
$$

Necessary conditions

If L exists then

1. (K,J) is compatible with (F,G)
2. (K,J) is independent

$$
\begin{tikzcd}
 & \mathbf{D} \arrow[ddl, bend right, "K"'] \arrow[ddr, bend left, "J"] \arrow[d, dashed, "L"] & \\
 & \mathbf{P} \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
\mathbf{A} \arrow[dr, "F"'] & \text{Proxy Pullback} & \mathbf{B} \arrow[dl, "G"] \\
 & \mathbf{C} &
\end{tikzcd}
$$

Compatibility

Compatibility

$$
\begin{tikzcd}
& D \arrow[dl, "K"'] \arrow[dr, "J"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
& C &
\end{tikzcd}
$$

Compatibility

D

$$
\begin{tikzcd}
 & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}
$$

Compatibility

D

$$\begin{tikzcd} & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\ A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd}$$

KD

JD

FKD

Compatibility

D

$KD \xrightarrow{a} \cdot$

$$\begin{tikzcd} & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\ A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd}$$

JD

FKD

Compatibility

$$D \xrightarrow{K^D a} \cdot$$

$$KD \xrightarrow{a} \cdot$$

$$\begin{tikzcd} & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\ A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd}$$

JD

FKD

Compatibility

$$D \xrightarrow{K^D a} \cdot$$

$$KD \xrightarrow{a} \cdot$$

$$\begin{tikzcd}
 & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}$$

$$JD \xrightarrow{JK^D a} \cdot$$

$$FKD$$

Compatibility

$$D \xrightarrow{K^D a} \cdot$$

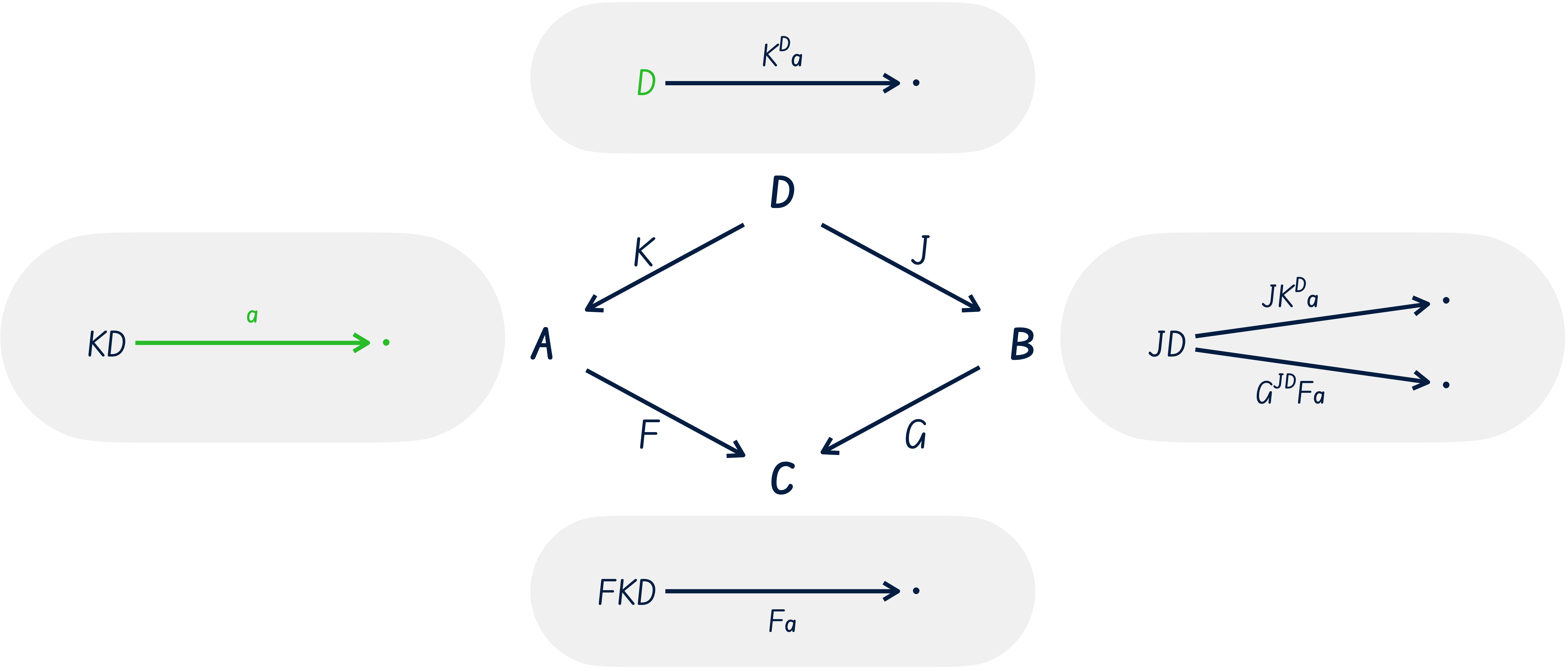
$$\begin{tikzcd}
 & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}$$

$$KD \xrightarrow{a} \cdot$$

$$JD \xrightarrow{JK^D a} \cdot$$

$$FKD \xrightarrow[Fa]{} \cdot$$

Compatibility



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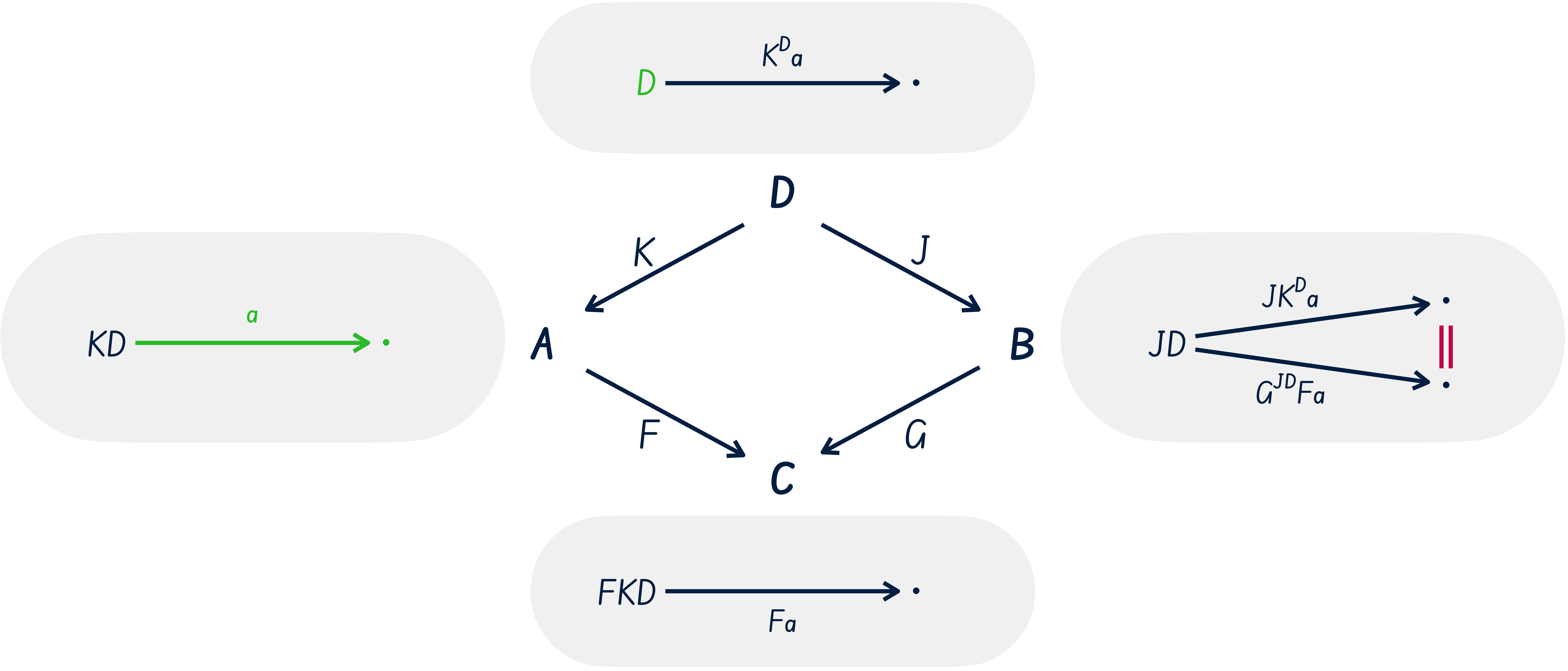
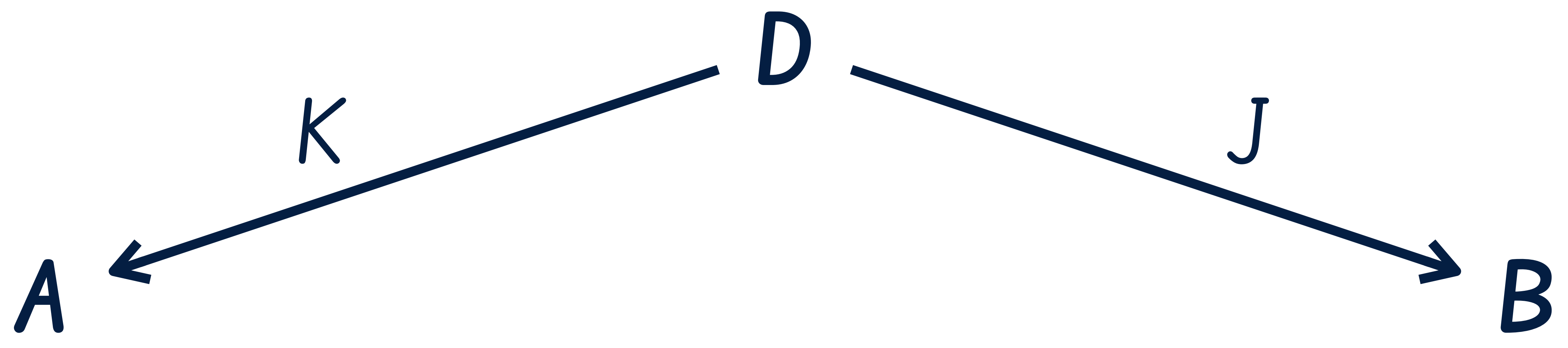


Independence

Independence



Independence

D

$$\begin{tikzcd} & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\ A & & B \end{tikzcd}$$

Independence

D

$$A \xleftarrow{K} D \xrightarrow{J} B$$

KD

JD

Independence

D

$$\begin{tikzcd}
 & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\
A & & B
\end{tikzcd}$$

$KD \xrightarrow{a_1} A_1$

JD

Independence

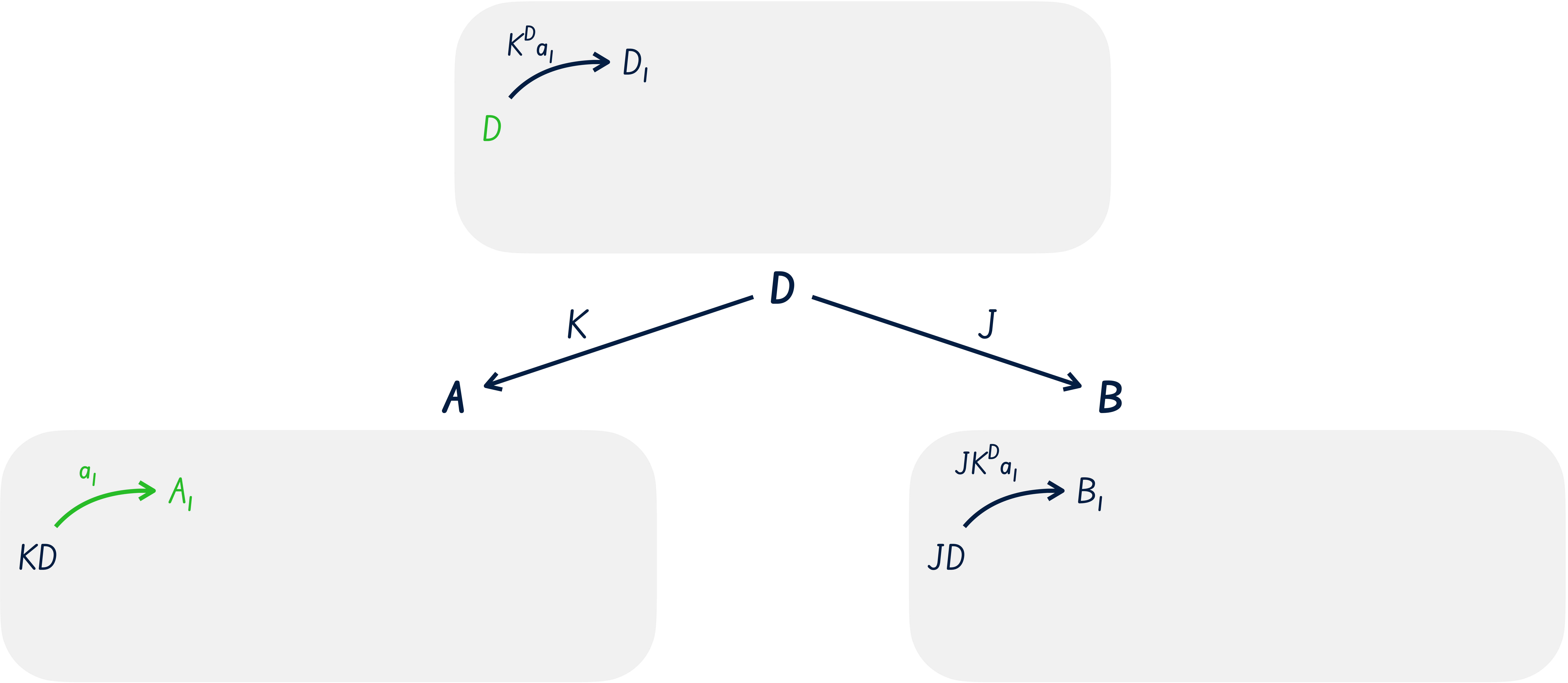
$$D \xrightarrow{K^D a_1} D_1$$

$$\begin{tikzcd} & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\ A & & B \end{tikzcd}$$

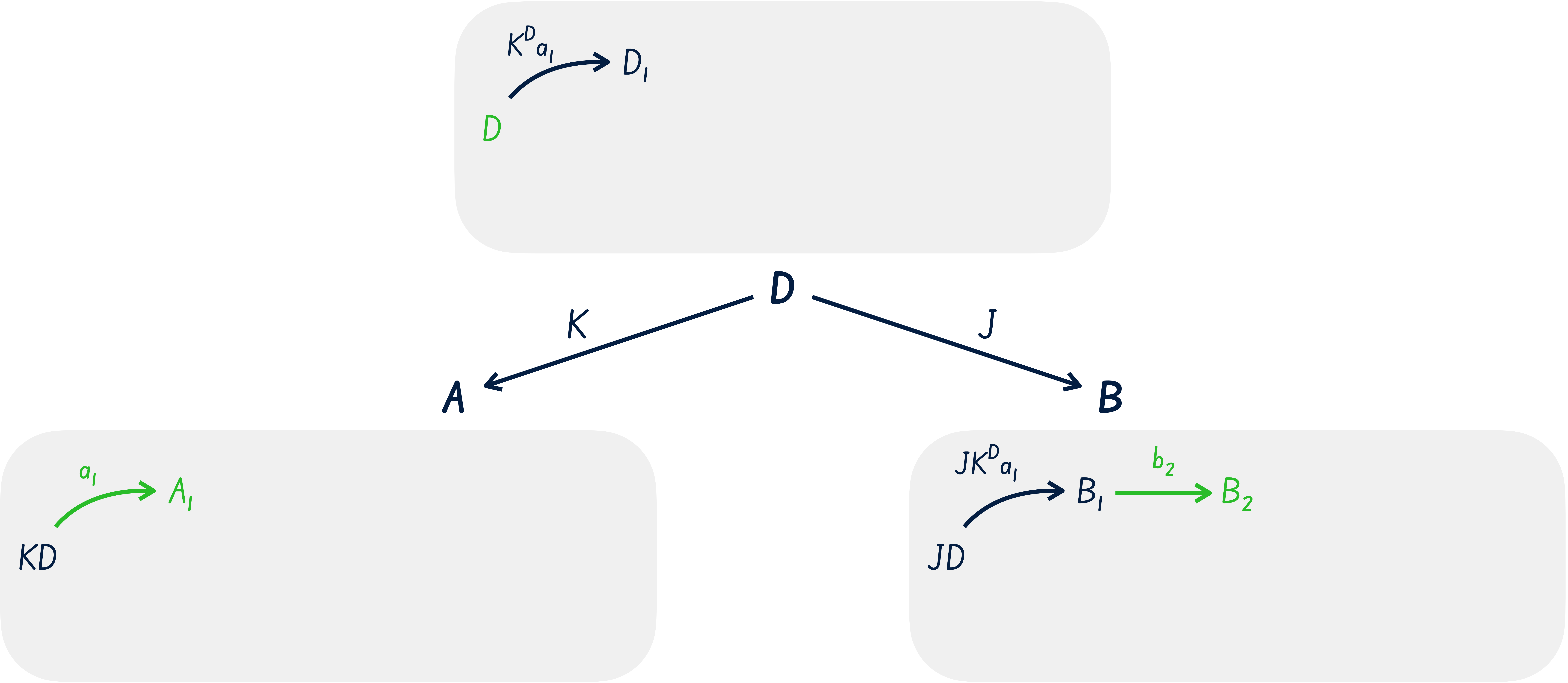
$$KD \xrightarrow{a_1} A_1$$

$$JD$$

Independence



Independence



Independence

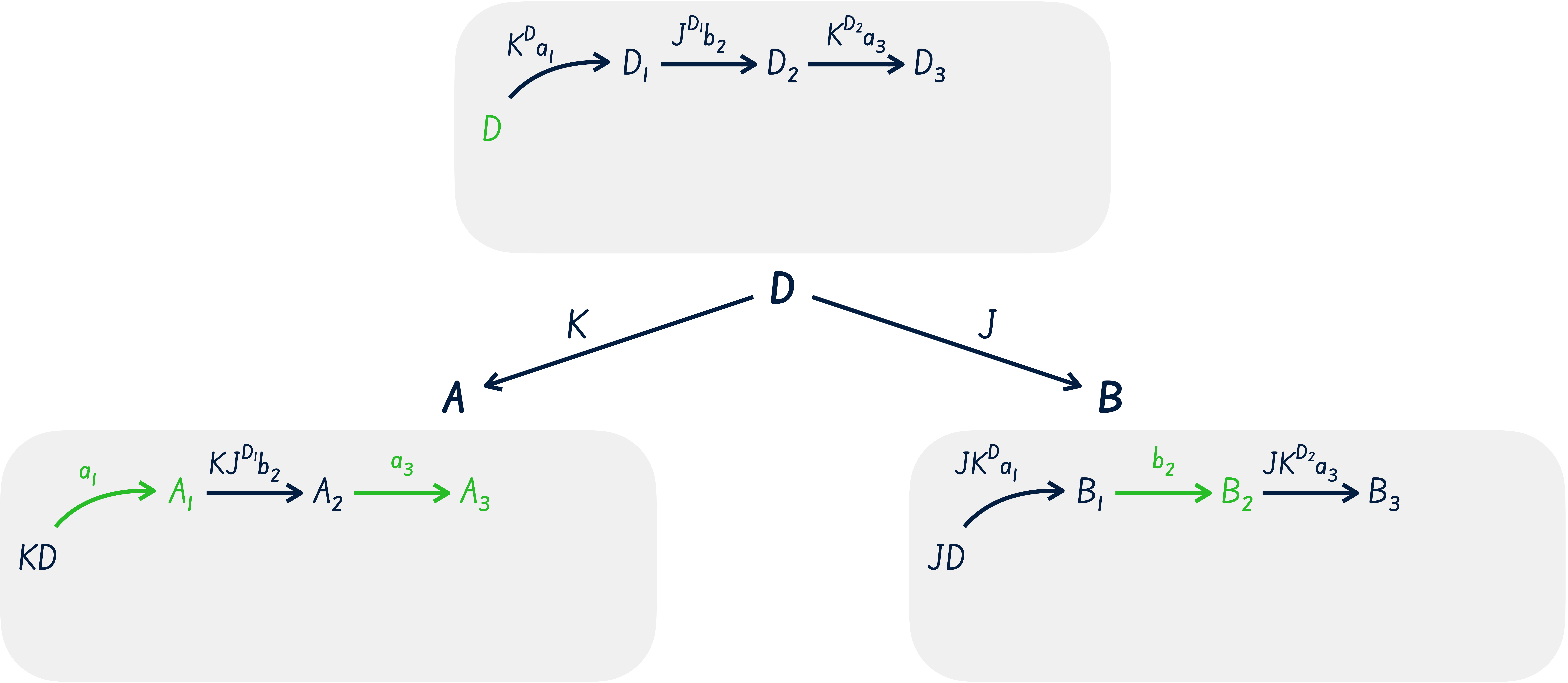
$$D \xrightarrow{K^D a_1} D_1 \xrightarrow{J^{D_1} b_2} D_2$$

$$\begin{tikzcd} & D \arrow[dl, "K"'] \arrow[dr, "J"] & \\ A & & B \end{tikzcd}$$

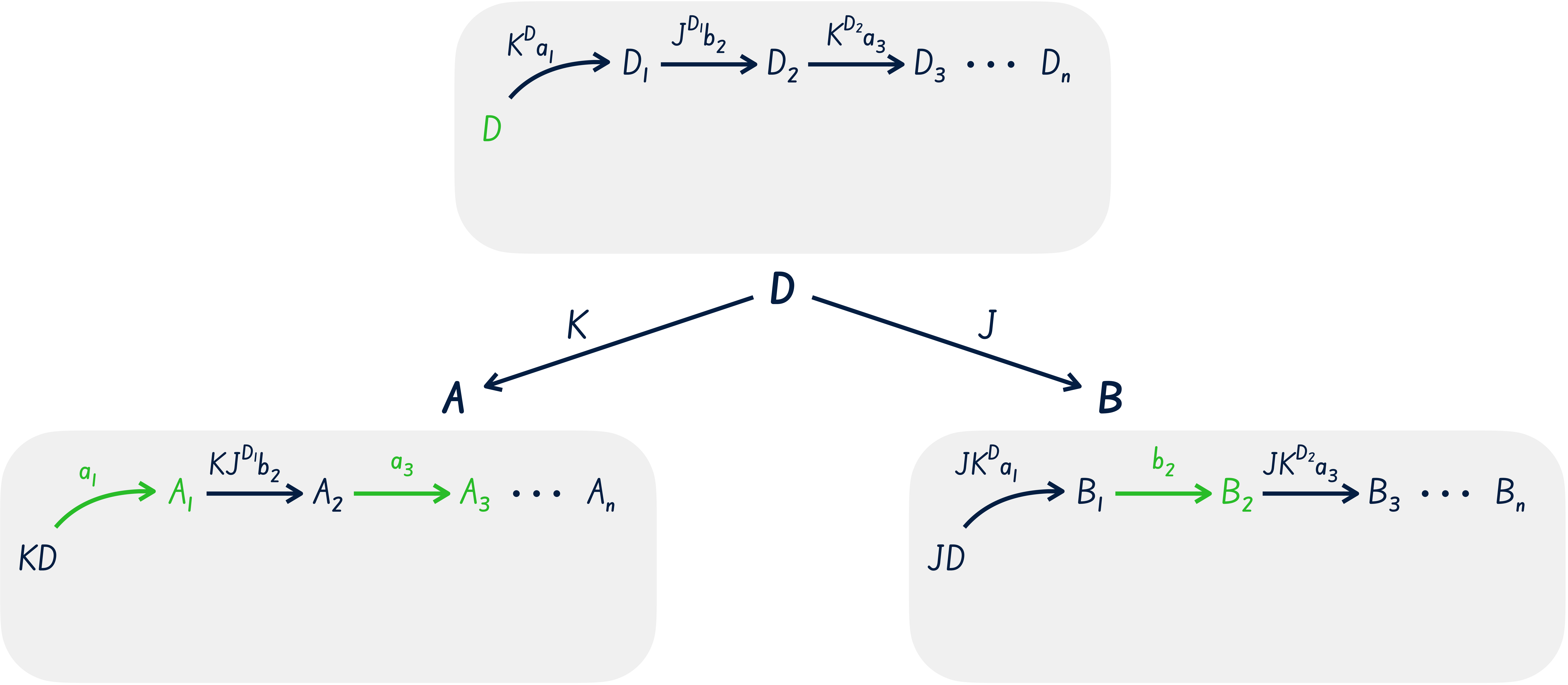
$$KD \xrightarrow{a_1} A_1 \xrightarrow{KJ^{D_1} b_2} A_2$$

$$JD \xrightarrow{JK^D a_1} B_1 \xrightarrow{b_2} B_2$$

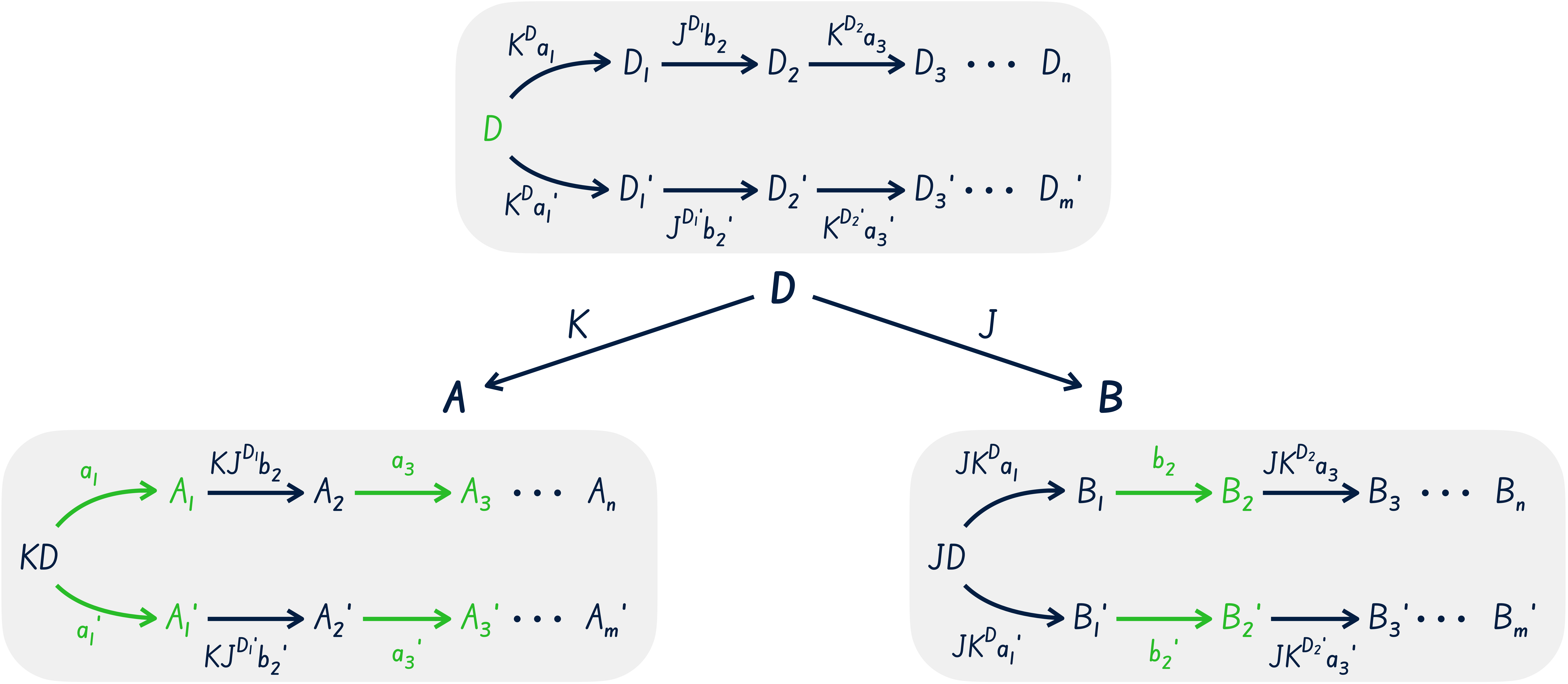
Independence



Independence



Independence



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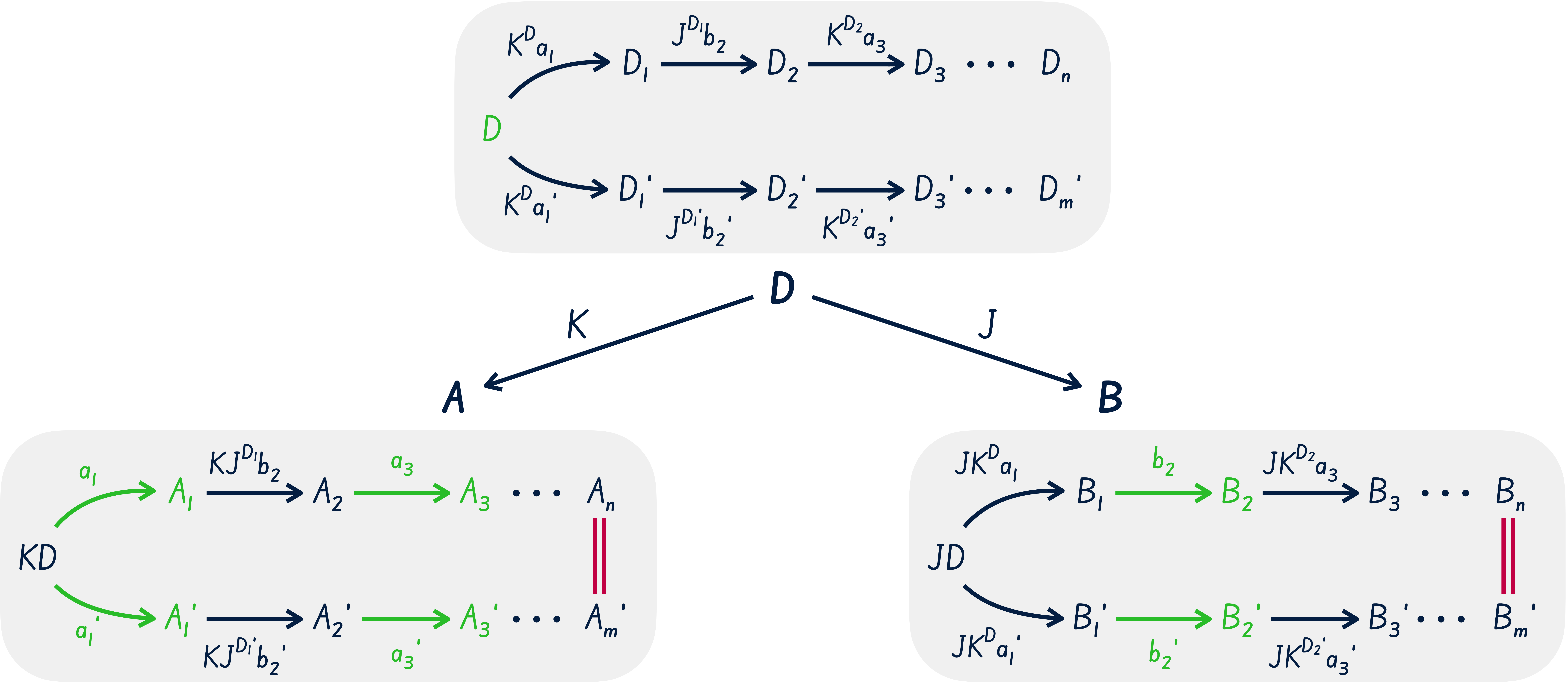
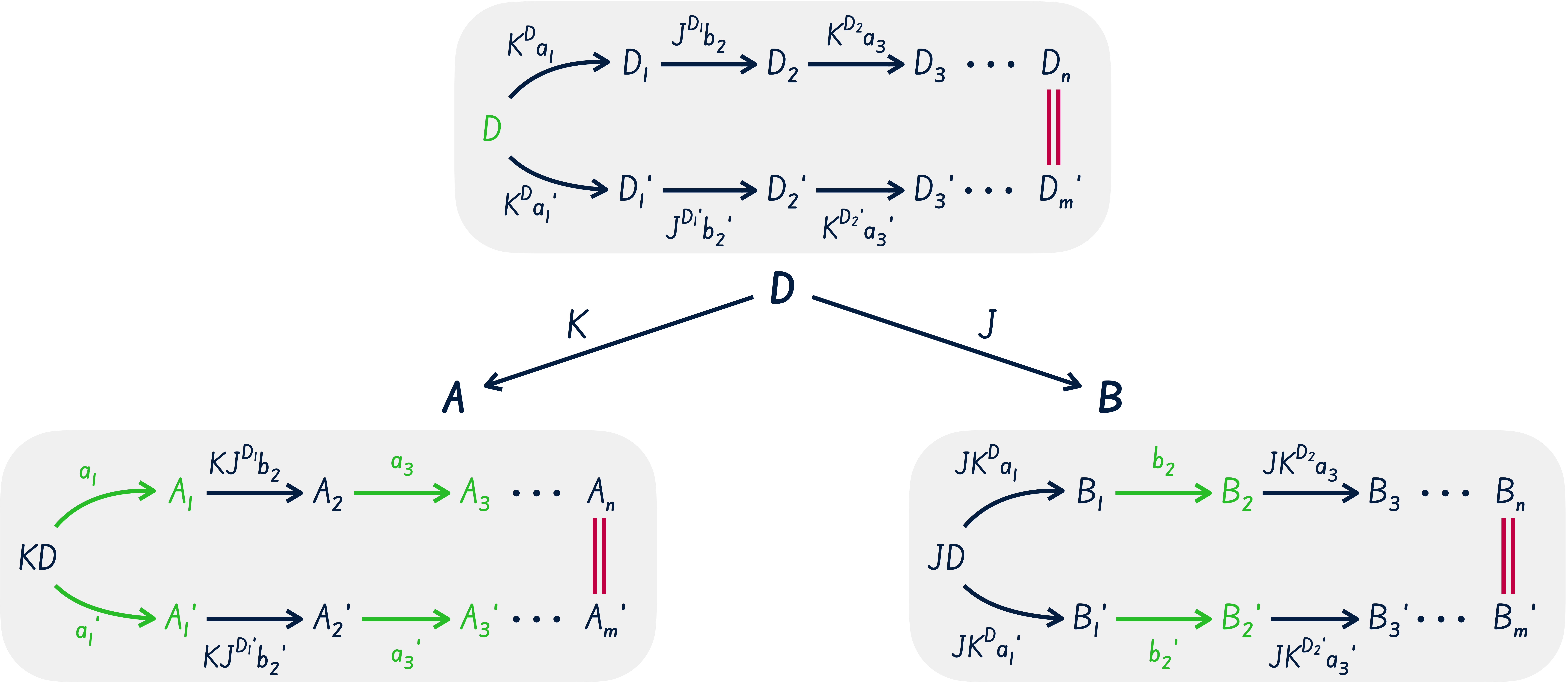


Independence



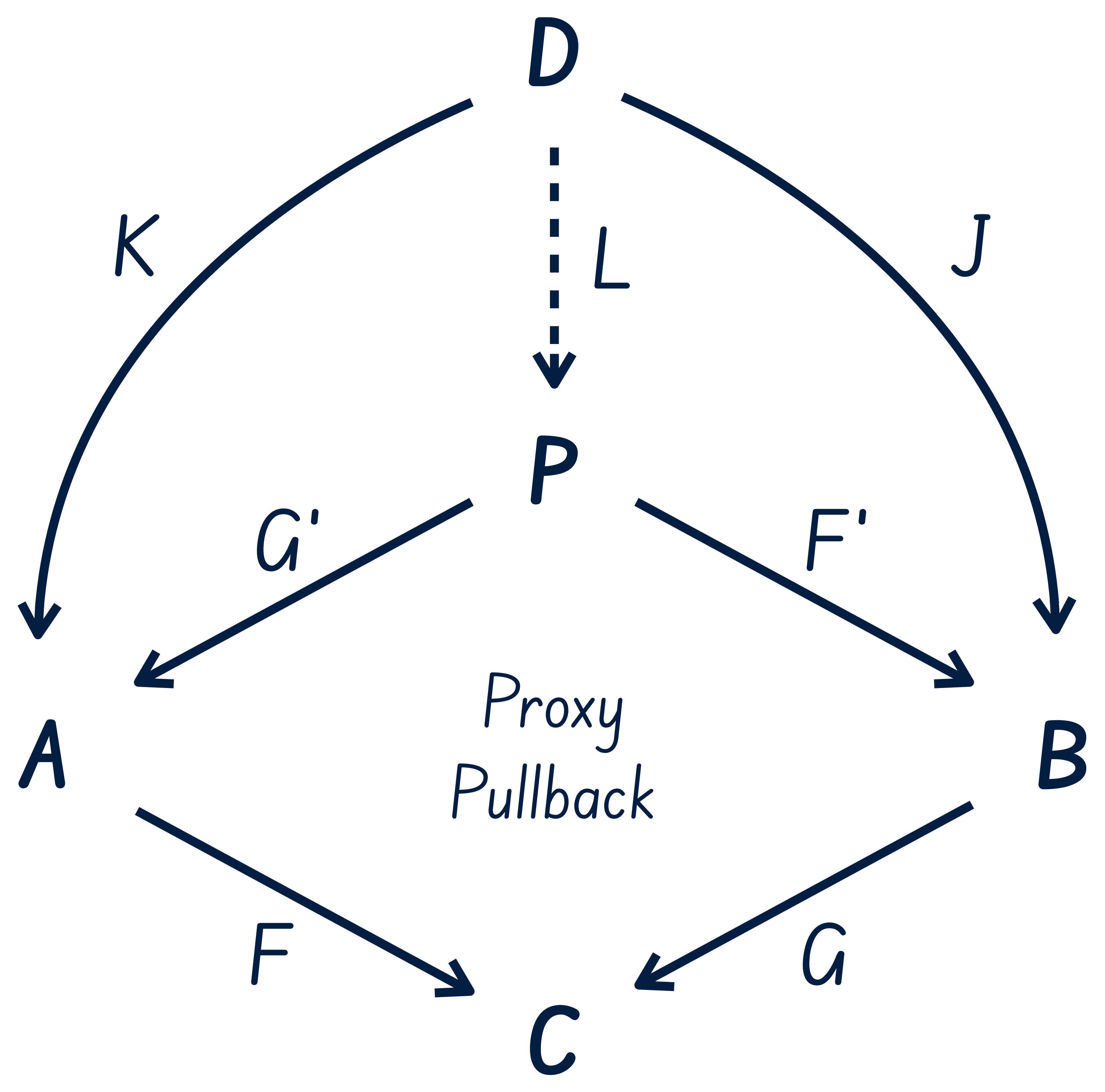
$$
\begin{tikzcd}
 & D \arrow[ddl, bend right, "K"'] \arrow[d, dashed, "L"] \arrow[ddr, bend left, "J"] & \\
 & P \arrow[dl, "G'"'] \arrow[dr, "F'"] \arrow[dd, phantom, "\text{Proxy Pullback}"] & \\
A \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\
 & C &
\end{tikzcd}
$$

Sufficient conditions

$$
\begin{tikzcd}
& D \arrow[ddl, bend right, "K"'] \arrow[d, dashed, "L"] \arrow[ddr, bend left, "J"] & \\
& P \arrow[dl, "G'"'] \arrow[dr, "F'"] & \\
A \arrow[dr, "F"'] & \text{Proxy Pullback} & B \arrow[dl, "G"] \\
& C &
\end{tikzcd}
$$

Sufficient conditions

(G',F') **sync minimal** $\iff$ L exists **for all** (K,J) independent and compatible with (F,G)



Conclusion

- *Sync-minimal* lens proxy pullbacks are universal amongst the *independent* and *compatible* lens spans

- This characterisation allowed a better understanding of when lens proxy pullbacks are real pullbacks

- Approach was inspired by Böhm and Simpson's treatment of pullback proxies in other categories

`mdimeglio.github.io`